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CONVERGENCE RATES RELATED TO THE STRONG LAW OF LARGE NUMBERS.(U)

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STRONG LAW OF LARGE NUMBERS

BY

JAMES ALLEN FILL

TECHNICAL REPORT NO. 14
DECEMBER 15, 1980

PREPARED UNDER CONTRACT
N00014-77-C-0306 (NR-042-373)
FOR THE OFFICE OF NAVAL RESEARCH

DEPARTMENT OF STATISTICS
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Also supported in part by National Science Foundation Grants SOC 72-05228 A04 and MCS 72-04364 A04 from University of Chicago

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ACKNOWLEDGMENTS

I wish to thank Michael Wichura, who has served as my advisor throughout the preparation of this dissertation. From formulation of the convergence rate problem through several revisions of its solution he has supplied unfailing guidance and enthusiastic support.

This thesis was typed using RUNOFF, a text-formatting program on the University of Chicago's DEC-20 system. Ronald Thisted's program SUPERSCRIPT was useful in preparing the mathematical type and designing the special symbols (including Greek letters). I thank Professor Thisted also for his generous computing advice.

I am grateful to the National Science Foundation and to the McCormick Foundation for their financial assistance.

Finally, I am indebted to Ellen, Jessica, and Erin Alderson for providing all the comforts of home.

TABLE OF CONTENTS

ABSTRACT

Let $X_1, X_2, \ldots$ be independent random variables with common distribution function F, zero mean, unit variance, and finite moment generating function, and with partial sums S_n. According to the strong law of large numbers,

$$p_m \equiv P\{\frac{S_n}{n} > c_n \text{ for some } n \geq m\}$$

decreases to 0 as m increases to ∞ when $c_n \equiv c > 0$. For general c_n's the Hewitt-Savage zero-one law implies that either $p_m = 1$ for every m or else $p_m \downarrow 0$ as $m \uparrow \infty$. Assuming the latter case, we consider here the problem of determining p_m up to asymptotic equivalence.

For constant c_n's the problem was solved by Siegmund (1975); in his case the rate of decrease depends heavily on F. In contrast, Strassen's (1965) solution for smoothly varying $c_n = o(n^{-2/5})$ is independent of F.

We complete the solution to the convergence rate problem by considering c_n's intermediate to those of Siegmund and Strassen. The rate in this case depends on an ever increasing number of terms in the Cramér series for F the more slowly c_n converges to zero.

CHAPTER I

INTRODUCTION AND SUMMARY

1.1. Basic assumptions and notation.

Throughout this work we suppose that $X_1, X_2, \ldots$ is a sequence of independent random variables with common distribution function F. Denote the random walk of partial sums by S:

$$S_n = \sum_{k=1}^{n} X_k, \; n \geq 0.$$

The distribution F is assumed to be standardized in the sense that

$$EX = 0, \text{ Var } X = 1$$

where, to facilitate notation, we have introduced another random variable X distributed according to F. Assume throughout that the moment generating function (mgf) $E\exp(\xi X)$ for F is finite for ξ in some neighborhood of 0 and write

(1.1.1) $\quad K(\xi) = \log(Ee^{\xi X})$

for the cumulant generating function (cgf). This assumption, which restricts attention to the so-called mgf case, is stronger than required for the more elementary results (for example, the laws of large numbers) discussed in this chapter. However, the loss in economy of assumption

is outweighed by the accompanying gain in ease of exposition. Furthermore, the main result (Theorem 2.1.1) of this work deals only with the mgf case.

Our main goal will be to estimate the probability of the event

$$\{\frac{S_n}{n} > c_n \text{ for some } n \geq m\}$$

when m is large for a specified sequence $c = (c_n)$ of positive numbers. It is natural to think of the sequence c as a "boundary" on the growth of the sequence (S_n/n) of sample means as the "time" n increases. Often it will be more convenient to deal either with the standardized process $(S_n/n^{1/2})$ or with the random walk S. The corresponding boundaries will be denoted as follows:

Process	Value at time n	Boundary	Value at time n
sample means	$\frac{S_n}{n}$	c	c_n
standardized	$\frac{S_n}{\sqrt{n}}$	Ψ	$\Psi(n) = \sqrt{n}c_n$
random walk	S_n	g	$g(n) = \sqrt{n}\Psi(n)$

We write Z for a standard normal random variable. $B = (B(t))_{t \geq 0}$ denotes Brownian motion.

We denote by $\lfloor x \rfloor$ the integer part, or "floor", of x, namely, the largest integer not exceeding x. Similarly, $\lceil x \rceil$, the "ceiling" of x, is the smallest integer at least as large as x.

If $k \geq 1$ is an integer, L_k denotes k iterations of the natural logarithm function $L \equiv \log$.

As usual, the relation $a(t) \approx b(t)$ means that $a(t) = O(b(t))$ and $b(t) = O(a(t))$.

1.2. Weak law of large numbers.

Although the present work concerns itself with convergence rates related to the strong law of large numbers, we begin with an examination of convergence rates related to the weak law. There are two reasons for this review: the weak-law results (1) provide motivation for, and (2) are used in the proofs of, the corresponding strong-law theorems.

The weak law of large numbers (WLLN) states that $S_m/m \to 0$ in probability as $m \to \infty$, i.e., that for any constant $c > 0$

$$(1.2.1) \qquad P\{|\frac{S_m}{m}| > c\} \to 0$$

as $m \to \infty$. Treating upper and lower tails separately, we can write (1.2.1) in the equivalent form

$$P\{\frac{S_m}{m} > c\} \to 0,$$

(1.2.1a)

$$P\{-\frac{S_m}{m} > c\} \to 0,$$

since

$$(1.2.2) \qquad P\{|\frac{S_m}{m}| > c\} = P\{\frac{S_m}{m} > c\} + P\{-\frac{S_m}{m} > c\}.$$

The convergence rate problem for the WLLN is to determine the left side of (1.2.1) up to a factor $(1 + o(1))$. In light of (1.2.2) this can be accomplished by estimating in the same way the single-tail probabilities in (1.2.1a). Since the random walk $(-S)$ satisfies the assumptions of Section 1.1 for S with F replaced by $P\{-X \le \cdot\}$, it is enough to deal with the upper tail probability $P\{S_m/m > c\}$.

A more general problem is to determine the asymptotic behavior of

$$(1.2.3) \qquad P\{\frac{S_m}{m} > c_m\}$$

for an arbitrary sequence of positive numbers c_m. In view of the central limit theorem (CLT) for S, it is convenient to express (1.2.3) in the standardized form

$$(1.2.3a) \qquad P\{\frac{S_m}{\sqrt{m}} > \Psi(m)\}$$

where we define

$$\Psi(m) = \sqrt{m}c_m.$$

Indeed, the CLT states that when $\Psi(m) \equiv \Psi_0$ is constant,

$$(1.2.4) \qquad P\{\frac{S_m}{\sqrt{m}} > \Psi_0\} \to P\{Z > \Psi_0\}.$$

The CLT is thus an invariance principle in the sense that the right side here is independent of F.

The Berry-Esséen theorem (see Feller, 1971, p. 542) bounds the error in approximating the left side of (1.2.4) by the right, uniformly in Ψ_0. As a particular consequence,

$$(1.2.5a) \qquad P\{\frac{S_m}{\sqrt{m}} > \Psi(m)\} \to 0$$

if and only if

$$(1.2.5b) \qquad \Psi(m) \to \infty .$$

The general problem of convergence rates related to the WLLN is to determine (1.2.3) up to a factor $(1 + o(1))$ when (1.2.5) is in force.

Return to the case $\Psi(m) \equiv cm^{1/2}$ of (1.2.1). Any unified theory for handling this case must require c to be small in some sense. For example, if S has symmetric Bernoulli components, i.e., if X assumes the values ± 1 with probability $1/2$ each, then for any $c \geq 1$

$$(1.2.6) \qquad P\{\frac{S_m}{m} > c\} = 0 \quad \text{for every } m.$$

The identity (1.2.6) cannot hold for any c, on the other hand, unless F has compact support.

Making precise the condition that c be small, Bahadur and Ranga Rao (1960) solved the WLLN convergence rate problem. Their solution is contained in Section 3.4; see (3.4.12) with $\alpha = 0$. In marked contrast to the invariance principle (1.2.4), the rate of convergence in (1.2.5a) in this case depends heavily on F. In fact, different choices for F give different convergence rates for (1.2.5a) for <u>some</u> $c > 0$.

The case $\Psi(m) = o(m^{1/2})$ with $\Psi(m) \to \infty$, intermediate to the CLT case of constant Ψ and the WLLN case $\Psi(m) = cm^{1/2}$, was resolved by Cramér (1938):

$$(1.2.7) \qquad P\{\frac{S_m}{\sqrt{m}} > \Psi(m)\} = (1 + o(1))P\{Z > \Psi(m)\}\cdot \exp[\Psi^2(m)\frac{\Psi(m)}{\sqrt{m}}\lambda(\frac{\Psi(m)}{\sqrt{m}})].$$

Here

$$\lambda(\xi) = \Sigma_{k\geq 0}\ \lambda_k \xi^k$$

is a certain power series, the so-called Cramér series for F, which converges for ξ in a neighborhood of 0. For each k the coefficient λ_k depends on the moments of F of orders up to and including $k + 3$; for example,

$$\lambda_0 = \frac{1}{6}EX^3, \quad \lambda_1 = \frac{1}{24}EX^4 - \frac{1}{8} - \frac{1}{8}(EX^3)^2.$$

For a precise definition of λ, see (2.1.1).

For the normal tail probability on the right in (1.2.7) we have the standard estimate

(1.2.8) $$P\{Z > \Psi(m)\} \sim [\sqrt{2\pi}\Psi(m)]^{-1}\exp[-\tfrac{1}{2}\Psi^2(m)]$$

(as usual, $a_m \sim b_m$ means the same as $a_m = (1 + o(1))\, b_m$).
On a logarithmic scale the correction in (1.2.7) to the normal approximation becomes negligible:

$$\log P\{\frac{S_m}{\sqrt{m}} > \Psi(m)\} \sim -\tfrac{1}{2}\Psi^2(m).$$

Even on the probability scale of (1.2.7), the correction is unnecessary if Ψ does not grow too rapidly:

$$\Psi(m) = o(m^{1/6}) \text{ implies}$$

(1.2.9a) $$P\{\frac{S_m}{\sqrt{m}} > \Psi(m)\} \sim P\{Z > \Psi(m)\}.$$

If Ψ is allowed to increase somewhat more quickly, the correction requires only the constant term λ_0 from the Cramér series:

$$\Psi(m) = o(m^{1/4}) \text{ implies}$$

(1.2.9b) $$P\{\frac{S_m}{\sqrt{m}} > \Psi(m)\} \sim P\{Z > \Psi(m)\}\cdot\exp[\lambda_0\frac{\Psi^3(m)}{\sqrt{m}}].$$

The linear coefficient λ_1 enters next:

$$\Psi(m) = o(m^{3/10}) \text{ implies}$$

(1.2.9c) $$P\{\frac{S_m}{\sqrt{m}} > \Psi(m)\} \sim P\{Z > \Psi(m)\}\cdot\exp[\lambda_0\frac{\Psi^3(m)}{\sqrt{m}}]\cdot\exp[\lambda_1\frac{\Psi^4(m)}{m}].$$

In general, if $\Psi(m) \to \infty$ and $\Psi(m) = o(m^{1/2 - \eta})$ with $0 < \eta \leq 1/3$, then only the moments of F of orders up to and including $\lceil 1/\eta \rceil - 1$ need be known to identify the convergence rate (1.2.7).

The transitions in form from the CLT to Cramér's result and from Cramér's result to the WLLN solution are smooth. When Ψ is nearly constant, as in (1.2.9a), Cramér's result is an invariance principle of the same form as the CLT. When, at the other extreme, $\Psi(m)$ grows nearly as quickly as $m^{1/2}$, the convergence rate (1.2.7) depends heavily on F. The Bahadur - Ranga Rao result for $\Psi(m) = cm^{1/2}$ can be stated in the form

$$(1.2.10) \qquad P\{\frac{S_m}{\sqrt{m}} > \Psi(m)\} \sim (1 + \beta)P\{Z > \Psi(m)\}\exp[\Psi^2(m)\frac{\Psi(m)}{\sqrt{m}}\lambda(\frac{\Psi(m)}{\sqrt{m}})]$$

as $m \to \infty$, where β depends on c and heavily on F but vanishes in the limit as $c \to 0$. Thus (1.2.7) may be regarded as the limiting form of (1.2.10) when $c \to 0$.

1.3. Strong law of large numbers.

According to the strong law of large numbers (SLLN), $S_n/n \to 0$ with probability 1; equivalently, for any constant $c > 0$

$$(1.3.1) \qquad P\{|\frac{S_n}{n}| > c \text{ for some } n \geq m\} \downarrow 0 \text{ as } m \uparrow \infty.$$

Analogous to (1.2.2) is the decomposition

$$P\{\frac{S_n}{n} > c \quad \text{for some} \quad n \geq m\}$$

$$+ P\{-\frac{S_n}{n} > c \quad \text{for some} \quad n \geq m\}$$

$$(1.3.2) \qquad - P\{\frac{S_p}{p} > c \quad \text{for some} \quad p \geq m \quad \text{and} \quad -\frac{S_q}{q} > c \quad \text{for some} \quad q \geq m\}$$

for the probability in (1.3.1). In Section 4.3 we show that the last term in (1.3.2) is asymptotically negligible when compared to the sum of the first two terms. So we consider the one-sided version

$$(1.3.3) \qquad P\{\frac{S_n}{n} > c \quad \text{for some} \quad n \geq m\} \downarrow 0 \quad \text{as} \quad m \uparrow \infty$$

of (1.3.1).

A more general problem is to determine the *asymptotic behavior of*

$$(1.3.4) \qquad p_m \equiv P\{\frac{S_n}{\sqrt{n}} > \Psi(n) \quad \text{for some} \quad n \geq m\}$$

for an arbitrary sequence of positive numbers $\Psi(n)$. No matter what the sequence Ψ,

$$(1.3.5) \qquad p_m \downarrow p \equiv P\{\frac{S_n}{\sqrt{n}} > \Psi(n) \quad \text{i.o.} \quad \text{as} \quad n \rightarrow \infty\}.$$

It follows from the Hewitt-Savage zero-one law (Feller, 1971, p. 124) that

$$(1.3.6) \qquad p = 0 \quad \text{or} \quad p = 1.$$

The case $p = 1$ is trivial from the convergence rate viewpoint, for then $p_m = 1$ for every m. The classification of boundaries Ψ

according to the dichotomy (1.3.6) is effected by the Kolmogorov - Petrovski - Erdős - Feller integral test (cf. Jain, Jogdeo, and Stout, 1975):

<u>KPEF INTEGRAL TEST</u>. If $0 < \psi \uparrow$, then

$$(1.3.7) \qquad p = {0 \atop 1} \text{ according as } \int^{\infty} \frac{\psi(t)}{t} e^{-\frac{1}{2}\psi^2(t)}\, dt \; {< \atop =} \; \infty. \quad \square$$

Note that the criterion (1.3.7) is, like its weak-law analogue (1.2.5), an invariance principle.

There are counterparts to (1.3.4-7) for Brownian motion: If

$$p_s \equiv P\{B(t) > g(t) \text{ for some } t \geq s\}$$

then

$$p_s \downarrow p \equiv P\{B(t) > g(t) \text{ i.o. as } t \to \infty\}$$

and (1.3.6) and the KPEF test hold; here

$$(1.3.8) \qquad g(t) = \sqrt{t}\,\psi(t).$$

In fact, (1.3.7) for S is most easily proved from (1.3.7) for B by showing that S can be closely approximated by B (cf. (3.1.6)).

In the interesting case that $p = 0$ in (1.3.5) we say that g is an <u>upper class boundary</u> for the random walk S and write $g \in U(S)$. (Otherwise g is a <u>lower class boundary</u> and $g \in L(S)$.) Define $U(B)$ and $L(B)$ analogously. If $\psi \uparrow$, the KPEF tests allow us

to write $g \in U$ indifferently for $g \in U(S)$ (for any S) or $g \in U(B)$. A similar comment applies to the notation $g \in L$.

The test (1.3.7) gives rise to the celebrated law of the iterated logarithm (LIL), which quite precisely describes the "interface" between U and L.

LAW OF THE ITERATED LOGARITHM. If

$$g(t) = [2t(L_2 t + \tfrac{3}{2}L_3 t + L_4 t + \cdots + L_{\rho-1} t + (1 + \beta)L_\rho t)]^{1/2} \tag{1.3.9}$$

with $\rho > 3$, then

$$g \in \begin{matrix} U \\ \\ L \end{matrix} \quad \text{according as} \quad \beta \begin{matrix} > \\ \\ \leq \end{matrix} 0. \quad \square \tag{1.3.10}$$

The general problem of convergence rates related to the SLLN is to determine p_m up to a factor $(1 + o(1))$ when $g \in U$. Let $T_m = \inf\{n\colon n \geq m,\ S_n > g(n)\}$, the inf of the empty set being $+\infty$. Then $T_m \geq m$, and

$$p_m = P\{S_n > g(n) \ \text{ for some } \ n \geq m\} = P\{T_m < \infty\}$$

admits the decomposition

$$\begin{aligned} p_m &= P\{T_m = m\} + P\{m < T_m < \infty\} \\ &= P\{S_m > g(m)\} + P\{m < T_m < \infty\}. \qquad (1.3.11) \end{aligned}$$

The convergence rate for the first term is known from studying the WLLN; the second is new. For a simple lower bound we have

(1.3.12) $\quad p_m \geq P\{S_m > g(m)\}$.

Siegmund (1975) used the relation (1.3.11), together with the Bahadur - Ranga Rao estimate for the first term and his own analysis of the second, to solve the convergence rate problem in the SLLN case $g(t) = ct$. Strassen (1965) solved the problem for boundaries $g \in U$ not too far from the $U \setminus L$ interface (1.3.9) -- roughly speaking, for $g(t) = o(t^{3/5})$ as $t \to \infty$. The major contribution of this work is to complete the solution to the convergence rate problem by bridging the gap between Strassen's boundaries and Siegmund's.

We set the stage by reviewing, in the next two sections, the results of Strassen and Siegmund.

1.4. Strassen's result.

We recall the omnibus restriction to the mgf case. Modulo a precise definition of the adjective "smooth", Strassen's result (1965, thm. 1.4) can be stated as follows:

THEOREM 1.4.1 (Strassen). If $g \in U$ has a smooth derivative, $0 < \psi \uparrow$, and $g(t) \leq t^{3/5 - \gamma}$ for some $\gamma > 0$, then

(1.4.1) $$p_m \sim J_m \equiv \int_m^\infty \frac{1}{\sqrt{2\pi}} \frac{g'(t)}{\sqrt{t}} e^{-\frac{1}{2}\psi^2(t)} dt. \quad \square$$

REMARK 1.4.2. (a) Strassen used an intricate argument to show that

(1.4.2) $$P\{B(t) > g(t) \text{ for some } t \geq s\} \sim J_s \quad \text{as} \quad s \to \infty$$

and used this, along with approximation of S by B via Skorohod embedding (see Breiman, 1968), to deduce his invariance principle (1.4.1). In light of (1.2.9a) one might expect that the restriction on the growth of g could be eased to $g(t) = o(t^{2/3})$. This can in fact be done (Theorem 3.2.1).

(b) Theorem 4.3 in Strassen (1965), a lemma to (1.4.1) credited by Strassen to F. Jonas, is in error. As noted by Sawyer (1972), the Skorohod embedding time for X may not have finite mgf even though X does. Theorem 3.2.1 repairs the proof of (and yields a result somewhat better than) Theorem 1.4.1 by using the dyadic quantile-transformation approximation of S by B due to Komlós, Major, and Tusnády (1975; 1976) instead of Skorohod embedding.

(c) If we assume $g(t)/t^{3/5} \downarrow$, then (compare the proof of Lemma 2.2.1(b))

$$g'(t) \approx \frac{\Psi(t)}{\sqrt{t}}.$$

In fact,

$$\frac{3}{5} \leq g'(t)/(\frac{\Psi(t)}{\sqrt{t}}) \leq 1.$$

Thus

$$p_m \approx \int_m^\infty \frac{\Psi(t)}{t} e^{-\frac{1}{2}\Psi^2(t)}\, dt,$$

which is the tail integral in the KPEF test (1.3.7). ▯

EXAMPLE 1.4.3. Define $g \in U$ at the $U \setminus L$ interface according to (1.3.9), with $\rho > 3$ and $\beta > 0$. Then

$$(1.4.3) \qquad P\{S_m > g(m)\} \sim [2\sqrt{\pi}(Lm)(L_2m)^2(L_3m) \cdots (L_{\rho-2}m)(L_{\rho-1}m)^{1+\beta}]^{-1}$$

which is of much smaller order of magnitude than

$$(1.4.4) \qquad p_m \sim [2\sqrt{\pi}\beta(L_{\rho-1}m)^{\beta}]^{-1}.$$

In contrast we shall see for Siegmund's boundaries and for those of Theorem 1.6.1 (and also for Strassen's when g is not too close to L) that

$$(1.4.5) \qquad p_m \approx P\{S_m > g(m)\}.$$

The extreme reluctance with which (1.4.4) tends to zero is a well-known phenomenon connected with the LIL. Were g only slightly smaller we would have $g \in L$ and hence $p_m = 1$ for every m. ▯

1.5. Siegmund's result.

In stating Siegmund's solution to the SLLN case $g(t) = ct$ we assume that $c > 0$ is sufficiently small. The criterion of smallness, detailed in Section 3.4, is the same as for the Bahadur - Ranga Rao WLLN result. We further assume that if F is a lattice distribution with span h, then c is a point in that lattice.

<u>THEOREM</u> 1.5.1 (Siegmund). If $g(t) = ct$ with $c > 0$ as above, then there is a constant $\gamma > 0$ for which

(1.5.1) $$p_m \sim (1 + \gamma)P\{S_m > g(m)\}. \quad \square$$

<u>REMARK</u> 1.5.2. (a) Siegmund determined the constant γ explicitly and remarked that

(1.5.2) $$\gamma \to 1 \quad \text{as} \quad c \to 0.$$

Nevertheless, for fixed c the constant γ, like the constant β and the series λ in (1.2.10), depends heavily on the component distribution F. So Siegmund's result, unlike Strassen's, is far from an invariance principle.

(b) Siegmund utilized the decomposition (1.3.11). In analyzing the second term he used the fundamental identity of sequential analysis, the large-deviation result of Bahadur and Ranga Rao, and some renewal-theoretic calculations. The same kind of approach is used in proving Lemma 2.4.1 to Theorem 2.1.1.

(c) For a generalization of Siegmund's theorem to linear boundaries g with nonzero intercept, see Theorem 3.4.1. $\square$

1.6. Completion of the solution to the convergence rate problem.

The solution to the convergence rate problem in the mgf case is completed by the following theorem (cf. Theorem 2.1.1), which overlaps somewhat with Strassen's:

THEOREM 1.6.1. If g has a smooth derivative and satisfies the monotonic growth conditions

(1.6.1) $$\frac{g(t)}{t^{1/2+\delta}} \uparrow$$

for some $\delta > 0$ and

(1.6.2) $$\frac{g(t)}{t} \downarrow 0,$$

then $g \in U$ and

(1.6.3) $$p_m \sim I_m \equiv \frac{g'(m)}{\sqrt{m}\Psi'(m)} \cdot P\{S_m > g(m)\}. \quad \square$$

REMARK 1.6.2. (a) That g belongs to U is an easy consequence of the KPEF test.

(b) The theorem's method of proof requires that g be kept away from the $U \setminus L$ interface and from linearity; hence the growth conditions (1.6.1) and (1.6.2).

(c) The rate of convergence of the factor $P\{S_m > g(m)\}$ to zero is given by Cramér's theorem (1.2.7). Thus in the present case the rate of convergence depends on F, but only through a (typically) finite number of cumulants of F.

(d) One might expect that a result like (1.4.1), but with a Cramér-like correction to the exponential factor, would hold. Indeed, we show in Lemma 3.2.4 that (1.6.3) can be recast in the form

$$(1.6.3a) \qquad p_m \sim \tilde{J}_m \equiv \int_m^\infty \frac{1}{\sqrt{2\pi}} \frac{g'(t)}{\sqrt{t}} e^{-\frac{1}{2}\Psi^2(t)} \exp[\Psi^2(t)\frac{\Psi(t)}{\sqrt{t}}\lambda(\frac{\Psi(t)}{\sqrt{t}})]\ dt.$$

In particular, (1.4.1) holds for $g(t) = o(t^{2/3})$.

(e) As in Siegmund's case, g increases rapidly enough that (1.4.5) holds (cf. Lemma 2.2.1(k)). □

EXAMPLE 1.6.3. The transitions from Theorems 1.4.1 to 1.6.1 to 1.5.1 are smooth. Let $g(t) = t^{1/2 + \delta}$ with $0 < \delta < 1/2$. Then

$$p_m \sim (1 + 1/(2\delta))P\{S_m > g(m)\}.$$

As δ tends to its lower limit 0, the factor $(1 + 1/(2\delta))$ tends to ∞. This is consistent with Example 1.4.3. As δ tends to its upper limit 1/2, $1/(2\delta)$ tends to 1, which is consistent with (1.5.2) in Remark 1.5.2(a). Furthermore, we have seen in Section 1.2 that the form of $P\{S_m > g(m)\}$ varies smoothly from the normal approximation to Cramér's theorem to the Bahadur - Ranga Rao result. □

1.7. A sketch of the proof.

In this section we present an outline of the proof of Theorem 1.6.1. For a precise statement and proof the reader is referred to Chapter 2.

We shall use the standard terminology " S crosses the boundary g at time m " to describe the event $\{S_m > g(m)\}$, although " S is above g at time m " is perhaps better. Notice that

$$\{S_{m-1} > g(m - 1),\ \ S_m > g(m),\ \ S_{m+1} > g(m + 1)\}$$

is included in this event, even though S does not "cross" from one side of g to the other.

The first step in the proof is to restrict to a finite interval $[m, v_m)$ those times at which S crosses g in the event $\{S_n > g(n)$ for some $n \geq m\}$ (whose probability is p_m). This is done by choosing v_m so as to satisfy two opposing constraints: (1) that the probability of a g -crossing after time v_m be negligible; and (2) that g be virtually linear over $[m, v_m)$. The "smooth derivative" condition guarantees that g' changes slowly enough to admit such a v_m.

The first criterion is made precise by using elementary subadditivity considerations and (1.3.12) to show that

(1.7.1) $\quad p_m \sim p_{m,v_m}$,

where

$$p_{m,v_m} \equiv P\{S_n > g(n) \text{ for some } m \le n < v_m\}.$$

Using the mean value theorem, we then trap the graph of g over the time-interval $[m, v_m)$ between approximating lower and upper lines $\underline{\ell}_m$ and $\overline{\ell}_m$, respectively, both passing through the point $(m, g(m))$ and having slope $(1 + o(1))\, g'(m)$ (see Figure 1.7.1). Writing ℓ_m indifferently for $\underline{\ell}_m$ or $\overline{\ell}_m$, we define

$$p_{m,v_m}(\ell_m) \equiv P\{S_n > \ell_m(n) \text{ for some } m \le n < v_m\}.$$

Clearly

$$(1.7.2) \qquad p_{m,v_m}(\overline{\ell}_m) \le p_{m,v_m} \le p_{m,v_m}(\underline{\ell}_m),$$

which reduces the problem to that of showing

$$(1.7.3) \qquad p_{m,v_m}(\ell_m) \sim I_m$$

with $\ell_m = \underline{\ell}_m$ or $\overline{\ell}_m$ and I_m defined in (1.6.3).

Let

$$p_m(\ell_m) \equiv P\{S_n > \ell_m(n) \text{ for some } n \ge m\}.$$

correspond to $p_{m,v_m}(\ell_m)$ as p_m corresponds to p_{m,v_m}. If $T_m \equiv \inf\{n\colon n \ge m,\ S_n > \ell_m(n)\}$ (as in Section 1.3, with g replaced by ℓ_m), then

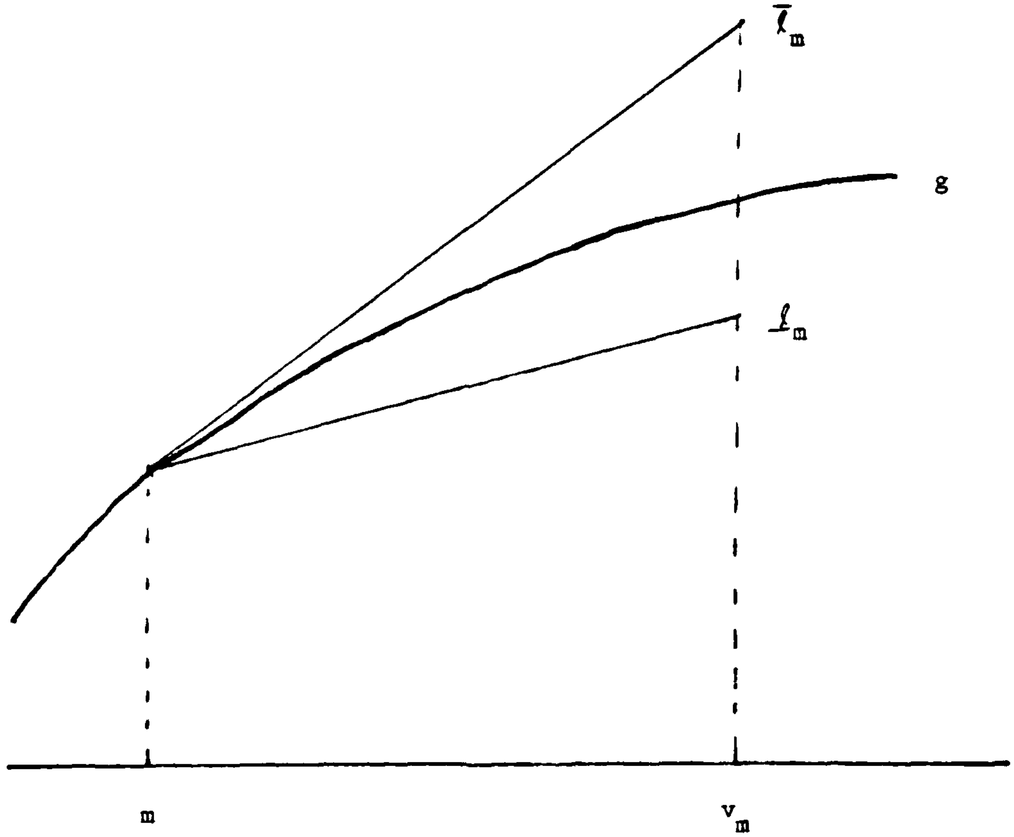

Fig. 1.7.1. g is trapped between approximating lines

$$p_m(\ell_m) = P\{T_m < \infty\}$$

$$= P\{S_m > \ell_m(m)\} + P\{m < T_m < \infty\}$$

$$= P\{S_m > g(m)\} + P\{m < T_m < \infty\}. \tag{1.7.4}$$

The first term can be estimated using Cramér's theorem (1.2.7). For the second we use Siegmund's techniques: the fundamental identity of sequential analysis, the appropriate large-deviation result (here, Cramér's theorem), and some renewal theory. The result is

$$p_m(\ell_m) \sim I_m. \tag{1.7.5}$$

From this we prove the analogue

$$p_{m,v_m}(\ell_m) \sim p_m(\ell_m) \tag{1.7.6}$$

to (1.7.1). Equation (1.7.3) then follows from (1.7.6) and (1.7.5), completing the proof of the theorem.

The heart of the proof lies with (1.7.5). It is enlightening to examine the proof of (1.7.5) under the simplifying assumption that $F = \Phi$, the standard normal distribution function.

In order to apply the fundamental identity of sequential analysis we first need to introduce the family of distributions associated with F through exponential tilting. In the special case $F = \Phi$ exponential tilting amounts to nothing more than a shift of location. Accordingly, let P_θ denote the probability under which X_1, X_2, ... are independent and normally distributed with mean θ and

unit variance. If

$$\ell_m(n) \equiv \alpha_m + \epsilon_m n$$

and T_m is redefined by

$$T_m \equiv \inf\{n: \ n \geq m, \ \ S_n > \alpha_m\},$$

then

$$(1.7.7) \qquad p_m(\ell_m) = P\{S_m > g(m)\} + P_{(-\epsilon_m)}\{m < T_m < \infty\}.$$

The idea now is to tilt from $P_{(-\epsilon_m)}$ to P_{ϵ_m}. There are two reasons for this. First, since $E_{\epsilon_m}(X) > 0$, we have by the SLLN the simplification

$$(1.7.8) \qquad \{m < T_m < \infty\} = \{T_m > m\} = \{S_m \leq \alpha_m\} \quad \text{a.s.} \quad P_{\epsilon_m}.$$

Second, letting $P_\theta^{(n)}$ denote the restriction of P_θ to the σ-field generated by $X_1, X_2, \ldots, X_n$, the Radon - Nikodym derivative of $P_{(-\epsilon_m)}^{(n)}$ with respect to $P_{\epsilon_m}^{(n)}$ assumes the particularly simple form

$$(1.7.9) \qquad \frac{dP_{(-\epsilon_m)}^{(n)}}{dP_{\epsilon_m}^{(n)}} = \exp(-2\epsilon_m S_n).$$

Thus

$$P_{(-\epsilon_m)}\{m < T_m < \infty\} = \textstyle\sum_{n>m} P_{(-\epsilon_m)}^{(n)}\{T_m = n\}$$

$$= \Sigma_{n>m} \int_{\{T_m = n\}} \exp(-2\epsilon_m S_n)\, dP^{(n)}_{\epsilon_m}$$

$$(1.7.10) \qquad = \int_{\{m < T_m < \infty\}} \exp(-2\epsilon_m S_{T_m})\, dP_{\epsilon_m},$$

demonstrating the fundamental identity of sequential analysis. Recalling (1.7.8),

$$P_{(-\epsilon_m)}\{m < T_m < \infty\} = \int_{\{S_m \leq \alpha_m\}} \exp(-2\epsilon_m S_{T_m})\, dP_{\epsilon_m}$$

$$(1.7.11) \qquad = \exp(-2\epsilon_m \alpha_m)\, P_{\epsilon_m}\{S_m \leq \alpha_m\}$$

$$\cdot \int \exp[-2\epsilon_m(S_{T_m} - \alpha_m)]\, d(P_{\epsilon_m}\{\cdot | S_m \leq \alpha_m\}).$$

The second factor on the right in (1.7.11) is

$$P_{\epsilon_m}\{S_m \leq \alpha_m\} = P\{S_m \leq \alpha_m - \epsilon_m m\}$$

$$= P\{-S_m \geq -\alpha_m + \epsilon_m m\}$$

$$(1.7.12) \qquad = P\{S_m > -\alpha_m + \epsilon_m m\},$$

making use of the symmetry and continuity of Φ. One can show that

$$\frac{1}{\sqrt{m}}(\epsilon_m m - \alpha_m) \sim 2\sqrt{m}\Psi'(m) \approx \Psi(m) \to \infty ;$$

hence (1.2.8) implies that (1.7.12) equals

$$(1 + o(1))(2\pi)^{-1/2}[2m\Psi'(m)]^{-1}\exp[-\tfrac{1}{2}m^{-1}(\epsilon_m m - \alpha_m)^2]$$

$$(1.7.13) \qquad = (1 + o(1))\exp(2\epsilon_m \alpha_m)\frac{\Psi(m)}{2m\Psi'(m)}P\{S_m > g(m)\}.$$

For general F, Cramér's theorem (1.2.7) would be needed to evaluate the final probability in (1.7.12).

Finally, the integrand in the third factor on the right in (1.7.11) is the exponential of a product of two factors: $\mathfrak{C}_m$, which equals $(1 + o(1))\, g'(m)$ and so (see Lemma 2.2.1(b) and (1.6.2)) tends to zero as $m \to \infty$, and $S_{T_m} - \alpha_m$, the amount by which S first overshoots the level α_m at or after time m. Using a renewal-theoretic result of Lorden (1970) one can show that this excess is of smaller order of magnitude than $1/\mathfrak{C}_m$ and from this that the integral on the right in (1.7.11) tends to 1 as $m \to \infty$.

Combining the results of our calculations,

$$(1.7.14) \qquad p_m(\ell_m) = P\{S_m > g(m)\}\left[1 + (1 + o(1))\frac{\Psi(m)}{2m\Psi'(m)}\right],$$

which can be reduced without difficulty to (1.7.5).

1.8. Extensions.

Theorem 1.6.1 is restated and proved in detail in Chapter 2. In Chapter 3 we give a correct proof of Strassen's theorem and a slight generalization of Siegmund's. Chapter 4 discusses whether $>$ can be changed to $\geq$ or vice versa in the definition

$$p_m = P\{S_n > g(n) \quad \text{for some} \quad n \geq m\}$$

without affecting the convergence rate and considers the case of two boundaries. In Chapter 5 we obtain a partial asymptotic expansion for

p_m in Strassen's case. The expansion given by this invariance principle contains a greater number of terms the nearer g is to the $U \setminus L$ interface and in fact forms a complete asymptotic series when g leaves the range of Theorem 1.6.1. We also get asymptotic upper bounds on the relative error in the approximation (1.6.3).

We conclude this summary by listing without comment four problems ripe for future research.

(1) Develop complete asymptotic expansions for p_m to extend Theorems 1.5.1 and 1.6.1.

(2) Increase the dimensions of both X ("state") and m ("time").

(3) Allow the components X_k to have unequal distributions.

(4) Treat the non-mgf case. In other words, what results can be salvaged when only finiteness of low-order moments is assumed?

CHAPTER II

COMPLETION OF THE SOLUTION TO THE CONVERGENCE RATE PROBLEM

In this chapter we more carefully state and prove Theorem 1.6.1, thereby completing the solution to the general problem of convergence rates related to the SLLN.

2.1. Statement of theorem.

Let $X, X_1, X_2, \ldots$ be independent random variables with common distribution function F, zero mean, unit variance, and moment generating function $E\exp(\xi X)$ finite for ξ in some neighborhood of 0, and put

$$S_n = \textstyle\sum_{k=1}^{n} X_k,\ n \geq 0.$$

Let

$$K(\xi) = \log(Ee^{\xi X})$$

denote the cumulant gererating function corresponding to F. The so-called Cramér series

$$\lambda(\xi) = \textstyle\sum_{k=0}^{\infty} \lambda_k \xi^k$$

for F is defined implicitly for ξ near 0 by

$$(2.1.1)\qquad \xi^3\lambda(\xi) = K(z) - z\xi + \tfrac{1}{2}\xi^2,$$
$$z\ (\equiv z(\xi))\ \text{given by}\ K'(z) = \xi.$$

Let $g\colon (0, \infty) \to (0, \infty)$ and write

$$g(t) = \sqrt{t}\,\psi(t).$$

Define

$$(2.1.2)\qquad p_m = P\{S_n > g(n) \text{ for some } n \geq m\}.$$

THEOREM 2.1.1. Suppose that as $t \uparrow \infty$

$$(2.1.3)\qquad \frac{g(t)}{t^{1/2+\delta}} \uparrow$$

for some $0 < \delta < 1/2$ and

$$(2.1.4)\qquad \frac{g(t)}{t} \downarrow 0.$$

If g is continuously differentiable and if for some $0 < r < 1$

$$(2.1.5)\qquad g'(u) \sim g'(t)$$
$$\text{when}\ t, u \to \infty\ \text{with}\ t \leq u \leq t[1 + 1/\psi^{2r}(t)],$$

then $g \in U$ and

$$(2.1.6)\qquad p_m \sim I_m \equiv \frac{g'(m)}{\sqrt{m}\,\psi'(m)} P\{S_m > g(m)\} \text{ as } m \to \infty. \quad \square$$

REMARK 2.1.2. (a) The conclusion (2.1.6) continues to hold if the various regularity conditions imposed on g are assumed to hold only for large t.

(b) The technical interpretation of (2.1.5) is that both the sup and inf of the sets

(2.1.7) $$\left\{\frac{g'(u)}{g'(t)} : t \leq u \leq t[1 + 1/\Psi^{2r}(t)]\right\}$$

tend to unity as $t \to \infty$. This assumption is implied by the condition

(2.1.8) $$g'(u) \sim g'(t) \quad \text{as} \quad u \sim t \to \infty,$$

which we interpret to mean that for any function s of t, if $s(t) \sim t$ as $t \to \infty$, then both the sup and inf of the sets

$$\left\{\frac{g'(u)}{g'(t)} : u \text{ is between } t \text{ and } s(t)\right\}$$

tend to unity as $t \to \infty$.

(c) The rate of convergence for $P\{S_m > g(m)\}$ is specified by (1.2.7-8). $\Box$

2.2. Some facts about the boundary.

The present section is reserved for a list of elementary properties of g resulting from the assumptions (2.1.3-5).

<u>LEMMA</u> 2.2.1. Let $g: (0, \infty) \to (0, \infty)$ be continuously differentiable and satisfy (2.1.3-5), and let $g(t) = t^{1/2}\Psi(t)$. Then

(a) $\frac{\Psi(t)}{t^{\delta}} \uparrow$ (whence $\Psi(t) \uparrow \infty$) and $\frac{\Psi(t)}{\sqrt{t}} \downarrow 0$;

(b) $(\frac{1}{2} + \delta)\frac{g(t)}{t} \le g'(t) \le \frac{g(t)}{t}$;

(c) $\Psi'(t) = t^{-1/2}(g'(t) - \frac{1}{2}\frac{g(t)}{t})$;

(d) $\delta\,\frac{\Psi(t)}{t} \le \Psi'(t) \le \frac{1}{2}\,\frac{\Psi(t)}{t}$;

(e) $\frac{d}{dt}(\frac{\Psi(t)}{\sqrt{t}}) = -\frac{1}{2}\,\frac{\Psi(t)}{t^{3/2}} + \frac{\Psi'(t)}{\sqrt{t}}$;

(f) $0 \le -\frac{d}{dt}(\frac{\Psi(t)}{\sqrt{t}}) \le (\frac{1}{2} - \delta)\frac{\Psi(t)}{t^{3/2}}$;

(g) $0 \le -\Psi^2(t)\frac{d}{dt}(\frac{\Psi(t)}{\sqrt{t}}) \le (\frac{1}{2} - \delta)\frac{\Psi^3(t)}{t^{3/2}} = o(\frac{\Psi^2(t)}{t})$ as $t \to \infty$;

(h) $\Psi(u) \sim \Psi(t)$ as $u \sim t \to \infty$

(i) $\Psi^2(u) - \Psi^2(t) \to 0$ when $t, u \to \infty$ with $t \le u \le t + 1$;

(j) $\Psi'(u) \sim \Psi'(t)$ when $t, u \to \infty$ as in (2.1.5);

(k) $2 \le \frac{g'(t)}{\sqrt{t}\Psi'(t)} \le 1 + \frac{1}{2\delta}$.

<u>PROOF</u>. (b) We have

$$g'(t) = \frac{d}{dt}(t \cdot \frac{g(t)}{t}) = \frac{g(t)}{t} + t \cdot \frac{d}{dt}(\frac{g(t)}{t}) \le \frac{g(t)}{t}$$

by (2.1.4). The first inequality follows similarly from (2.1.3).

(d) Combine (b) and (c).

(f) Combine (d) and (e).

(g) Use (f) and (a).

(h) If $t \le u$ then (a) yields

$$\Psi(t) \le \Psi(u) \le \left(\frac{u}{t}\right)^{1/2}\Psi(t) = (1 + o(1))\Psi(t);$$

the case $t > u$ is handled similarly.

(i) By (a), the mean value theorem, (d), and (h),

$$0 \le \Psi^2(u) - \Psi^2(t) \le \Psi^2(t + 1) - \Psi^2(t) \le \Psi(t + 1) \cdot \frac{\Psi(t)}{t}$$

$$\le (1 + o(1))\frac{\Psi^2(t)}{t} = o(1).$$

(j) Use (c), (b), and (h) and (2.1.5).

(k) Use (c) and (b). $\square$

In Sections 2.3 through 2.6 we prove Theorem 2.1.1 following the outline of Section 1.7.

2.3. Restriction of g -crossing times to a finite interval.

Throughout this section "fact (·)" refers to part (·) of Lemma 2.2.1.

Recall that we wish to define $v_m > m$ in such a way that

(2.3.1) $\quad p_{v_m} = o(p_m) \quad \text{as} \quad m \to \infty$

but g is virtually linear over $[m, v_m)$. As it turns out, an appropriate choice is

(2.3.2) $\quad v_m = \lfloor m(1 + 1/\Psi^{2r}(m)) \rfloor$.

Fact (a) implies that

(2.3.3) $\quad v_m \sim m$

and (2.1.5) yields

(2.3.4) $\quad g'(t) \sim g'(m) \quad \text{when} \quad m, t \to \infty \quad \text{with} \quad m \le t \le v_m$.

LEMMA 2.3.1. If v_m is defined by (2.3.2), then (2.3.1) holds.

PROOF. By subadditivity

(2.3.5) $\quad p_m \le \sum_{n \ge m} P\{S_n > g(n)\};$

we'll replace m by v_m in (2.3.5) to get an asymptotic upper bound on p_{v_m}. By Cramér's theorem,

$$P\{S_n > g(n)\} \sim [\sqrt{2\pi}\Psi(n)]^{-1}\exp\{-\tfrac{1}{2}\Psi^2(n)[1 - 2\tfrac{\Psi(n)}{\sqrt{n}}\lambda(\tfrac{\Psi(n)}{\sqrt{n}})]\}$$

$$(2.3.6) \qquad \sim [\sqrt{2\pi}\tilde{\Psi}(n)]^{-1}\exp\{-\tfrac{1}{2}\tilde{\Psi}^2(n)\},$$

where

$$(2.3.7) \qquad \tilde{\Psi}(t) \equiv \{\Psi^2(t)[1 - 2\frac{\Psi(t)}{\sqrt{t}}\lambda(\frac{\Psi(t)}{\sqrt{t}})]\}^{1/2}.$$

Note

$$(2.3.8) \qquad \tilde{\Psi}(t) \sim \Psi(t) \quad \text{as} \quad t \to \infty .$$

Also, as $t \to \infty$

$$\begin{aligned} 2\tilde{\Psi}(t)\tilde{\Psi}'(t) &= 2\Psi(t)\Psi'(t)[1 - 2\frac{\Psi(t)}{\sqrt{t}}\lambda(\frac{\Psi(t)}{\sqrt{t}})] \\ &\quad - 2\Psi^2(t) \cdot \frac{d}{dt}(\frac{\Psi(t)}{\sqrt{t}}) \cdot [\lambda(\frac{\Psi(t)}{\sqrt{t}}) + \frac{\Psi(t)}{\sqrt{t}}\lambda'(\frac{\Psi(t)}{\sqrt{t}})] \\ &= (1 + o(1))2\Psi(t)\Psi'(t) \\ &\quad - (1 + o(1))2\lambda_0\Psi^2(t) \cdot \frac{d}{dt}(\frac{\Psi(t)}{\sqrt{t}}) \\ &= (1 + o(1))2\Psi(t)\Psi'(t) \end{aligned}$$

by facts (d) and (g), so that

$$(2.3.9) \qquad \tilde{\Psi}'(t) \sim \Psi'(t)$$

Hence when $t, u \to \infty$ with $t \leq u \leq t + 1$

$$(2.3.10) \qquad 0 \leq \tilde{\Psi}^2(u) - \tilde{\Psi}^2(t) \leq (1 + o(1))\frac{\Psi^2(t)}{t} = o(1)$$

(cf. proof of fact (i)). From (2.3.5-10) and facts (h) and (d) follows

$$\sqrt{2\pi}p_m \leq (1 + o(1)) \textstyle\sum_{n\geq m} [\tilde{\Psi}(n)]^{-1} \exp[-\tfrac{1}{2}\tilde{\Psi}^2(n)]$$

$$= (1 + o(1)) \int_m^\infty [\tilde{\Psi}(t)]^{-1} \exp[-\tfrac{1}{2}\tilde{\Psi}^2(t)]\, dt$$

$$\leq (1 + o(1))\, \delta^{-1} \int_m^\infty \frac{t}{\tilde{\Psi}^2(t)} \exp[-\tfrac{1}{2}\tilde{\Psi}^2(t)]\, \tilde{\Psi}'(t)\, dt$$

$$= O(\int_m^\infty \tilde{\Psi}^{1/\delta - 2}(t) \exp[-\tfrac{1}{2}\tilde{\Psi}^2(t)]\, \tilde{\Psi}'(t)\, dt).$$

But

$$\int_m^\infty \tilde{\Psi}^{1/\delta - 2}(t) \exp[-\tfrac{1}{2}\tilde{\Psi}^2(t)]\, \tilde{\Psi}'(t)\, dt$$

$$= \int_{\tilde{\Psi}(m)}^\infty u^{1/\delta - 2}\, e^{-\frac{1}{2}u^2}\, du$$

$$\sim \tilde{\Psi}^{1/\delta - 3}(m) \exp[-\tfrac{1}{2}\tilde{\Psi}^2(m)]$$

so

$$(2.3.11) \qquad p_m = O(\tilde{\Psi}^{1/\delta - 3}(m) \exp[-\tfrac{1}{2}\tilde{\Psi}^2(m)]).$$

Now by the mean value theorem, (2.3.8), (2.3.9), facts (j) and (d), and (2.3.2),

$$\tilde{\Psi}^2(v_m) - \tilde{\Psi}^2(m) \geq (1 + o(1)) 2\delta \frac{\tilde{\Psi}^2(m)}{m}(v_m - m)$$

$$= (1 + o(1)) 2\delta \tilde{\Psi}^{2(1 - r)}(m)$$

and thus (2.3.11) (with m replaced by v_m) easily yields

$$p_{v_m} = o([\sqrt{2\pi}\tilde{\Psi}(m)]^{-1} \exp[-\tfrac{1}{2}\tilde{\Psi}^2(m)])$$

(2.3.12) $\qquad = o(P\{S_m > g(m)\})$.

To complete the proof of Lemma 2.3.1 use the obvious bound

(2.3.13) $\quad p_m \geq P\{S_m > g(m)\}$. $\square$

Define

(2.3.14) $\quad p_{m,v_m} = P\{S_n > g(n) \text{ for some } m \leq n < v_m\}$;

clearly,

(2.3.15) $\quad p_{m,v_m} \leq p_m \leq p_{m,v_m} + p_{v_m}$.

We therefore immediately obtain

<u>COROLLARY</u> 2.3.2. With p_m given by (2.1.2) and p_{m,v_m} by (2.3.14),

(2.3.16) $\quad p_m \sim p_{m,v_m}$ as $m \to \infty$. $\square$

2.4. <u>Linearization of the boundary</u>.

We define here straight lines $\underline{\ell}_m$ and $\overline{\ell}_m$, both passing through the point $(m, g(m))$, which well approximate g over the interval $[m, v_m]$ (Fig. 1.7.1). The definition of $\underline{\ell}_m$ (respectively, $\overline{\ell}_m$) together with the mean value theorem will imply that this line minorizes (majorizes) g on $[m, v_m]$.

The slope $\underline{\epsilon}_m$ (respectively, $\overline{\epsilon}_m$) of $\underline{\ell}_m$ ($\overline{\ell}_m$) is defined to be the minimum (maximum) value of g' over the interval $[m, v_m]$. We now treat both lines at once by writing, for example, ℓ_m indifferently for $\underline{\ell}_m$ or $\overline{\ell}_m$. By (2.3.4)

(2.4.1) $\quad \epsilon_m \sim g'(m)$ as $m \to \infty$.

The y-intercept of ℓ_m is

(2.4.2) $\quad \alpha_m = g(m) - m\epsilon_m$.

In analogy with (2.1.2) and (2.3.14) define

(2.4.3) $\quad p_m(\ell_m) = P\{S_n > \ell_m(n) \text{ for some } n \geq m\}$

and

(2.4.4) $\quad p_{m,v_m}(\ell_m) = P\{S_n > \ell_m(n) \text{ for some } m \leq n < v_m\}$.

The key to analyzing (2.4.3) is the following lemma, to be proved in the next section.

<u>LEMMA</u> 2.4.1. Let ℓ_m denote the straight line

(2.4.5) $\quad \ell_m(t) = \alpha_m + \epsilon_m t$

and define $p_m(\ell_m)$ by (2.4.3). If $\epsilon_m > 0$ satisfies

(2.4.6) $\quad \epsilon_m \to 0$

and

(2.4.7) $\quad \sqrt{m}\epsilon_m \to \infty,$

and if

(2.4.8) $\quad \limsup_{m \to \infty} \frac{|\alpha_m|}{\epsilon_m m} < 1,$

then

(2.4.9) $\quad p_m(\ell_m) \sim \frac{2\epsilon_m}{\epsilon_m - \frac{\alpha_m}{m}} \cdot P\{S_m > \ell_m(m)\} \quad \text{as} \quad m \to \infty. \quad \square$

REMARK 2.4.2. The conditions (2.4.6-7) demand that ϵ_m tend to 0, but not too quickly. Assumption (2.4.8) requires, loosely speaking, that the limiting proportional contribution of the constant term to the value at $t = m$ of either $\ell_m(t) = \alpha_m + \epsilon_m t$ or $\tilde{\ell}_m(t) = -\alpha_m + \epsilon_m t$ (which arose in (1.7.12) for the example $F = \Phi$) is less than 1/2. It follows from (2.4.8-9) that

$$p_m(\ell_m) \approx P\{S_m > \ell_m(m)\}. \quad \square$$

According to (2.1.3-4), (2.4.1-2), and Lemma 2.2.1(b), $\ell_m = \underline{\ell}_m$ or $\overline{\ell}_m$ satisfies the assumptions of the lemma. Recall that

(2.4.10) $\quad \ell_m(m) = g(m).$

Furthermore, by (2.4.2), (2.4.1), and Lemma 2.2.1(b)-(c)

$$\epsilon_m - \frac{\alpha_m}{m} = 2[\epsilon_m - \frac{1}{2}\frac{g(m)}{m}]$$

$$= 2[(1 + o(1))g'(m) - \frac{1}{2}\frac{g(m)}{m}]$$

$$= (1 + o(1))2[g'(m) - \frac{1}{2}\frac{g(m)}{m}]$$

$$(2.4.11) \qquad = (1 + o(1))2\sqrt{m}\psi'(m).$$

Combining (2.4.1) and (2.4.9-11) we get

$$(2.4.12) \qquad p_m(\ell_m) \sim \frac{g'(m)}{\sqrt{m}\psi'(m)} P\{S_m > g(m)\} = I_m \quad \text{as} \quad m \to \infty,$$

completing the analysis of (2.4.3).

In Lemma 2.4.3 below we'll prove the analogue

$$(2.4.13) \qquad p_{v_m}(\ell_m) \equiv P\{S_n > \ell_m(n) \quad \text{for some} \quad n \geq v_m\} = o(p_m(\ell_m))$$

to (2.3.1). As an immediate corollary (cf. Corollary 2.3.2),

$$(2.4.14) \qquad p_{m,v_m}(\ell_m) \sim p_m(\ell_m) \quad \text{as} \quad m \to \infty.$$

We combine (2.4.14) and (2.4.12) to obtain the rate of convergence for (2.4.4):

$$(2.4.15) \qquad p_{m,v_m}(\ell_m) \sim I_m \quad \text{as} \quad m \to \infty.$$

Moreover, since $g(n)$ is trapped between $\underline{\ell}_m(n)$ and $\overline{\ell}_m(n)$ we have

$$(2.4.16) \qquad p_{m,v_m}(\overline{\ell}_m) \leq p_{m,v_m} \leq p_{m,v_m}(\underline{\ell}_m).$$

The main result (2.1.6) follows from (2.3.16) and (2.4.15-16).

The remainder of this section is devoted to the following corollary to Lemma 2.4.1.

LEMMA 2.4.3. With $\ell_m = \underline{\ell}_m$ or $\overline{\ell}_m$, (2.4.13) holds.

PROOF. To begin the proof of (2.4.13) we apply Lemma 2.4.1 to the left side to yield

$$(2.4.17) \qquad p_{v_m}(\ell_m) \sim \frac{2\epsilon_m}{\epsilon_m - \frac{\alpha_m}{v_m}} P\{S_{v_m} > \ell_m(v_m)\},$$

recalling (2.3.3) to verify the hypotheses (2.4.7-8). In light of (2.3.3) and (2.4.8), the first factor on the right in (2.4.17) asymptotes to the first factor on the right in (2.4.9), or, in the present context, to the first factor in I_m (recall the proof of (2.4.12)). Moreover, by Cramér's result

$$P\{S_{v_m} > \ell_m(v_m)\}$$

$$(2.4.18) \qquad \sim [\sqrt{2\pi}\Psi_m(v_m)]^{-1}\exp\{-\tfrac{1}{2}\Psi_m{}^2(v_m)[1 - 2\frac{\Psi_m(v_m)}{\sqrt{v_m}}\lambda(\frac{\Psi_m(v_m)}{\sqrt{v_m}})]\};$$

here we have written

$$(2.4.19) \qquad \ell_m(t) = \sqrt{t}\Psi_m(t)$$

so that $\Psi_m(m) = \Psi(m)$ and

$$\Psi_m(v_m) = v_m{}^{-1/2}\ell_m(v_m)$$

$$= v_m^{-1/2}(\alpha_m + \epsilon_m v_m)$$

$$= (1 + o(1))m^{-1/2}(\alpha_m + \epsilon_m m)$$

(2.4.20) $$= (1 + o(1))\Psi(m).$$

Put

(2.4.21) $$\tilde{\Psi}_m(t) = \{\Psi_m^{\,2}(t)[1 - 2\frac{\Psi_m(t)}{\sqrt{t}}\lambda(\frac{\Psi_m(t)}{\sqrt{t}})]\}^{1/2};$$

then

(2.4.22) $$P\{S_{v_m} > \ell_m(v_m)\} \sim [\sqrt{2\pi}\Psi(m)]^{-1}\exp[-\tfrac{1}{2}\tilde{\Psi}_m^{\,2}(v_m)].$$

But with

(2.4.23) $$f(\xi) = \xi^2[1 - 2\xi\lambda(\xi)] \sim \xi^2 \quad \text{as} \quad \xi \to 0,$$

so that

(2.4.24) $$f'(\xi) = 2\xi[1 - 2\xi\lambda(\xi)] - 2\xi^2[\lambda(\xi) + \xi\lambda'(\xi)] \sim 2\xi \quad \text{as} \quad \xi \to 0,$$

we have

$$\tilde{\Psi}_m^{\,2}(v_m) - \tilde{\Psi}_m^{\,2}(m) = v_m f(\frac{\Psi_m(v_m)}{\sqrt{v_m}}) - mf(\frac{\Psi_m(m)}{\sqrt{m}})$$

$$= -v_m[f(\frac{\Psi_m(m)}{\sqrt{m}}) - f(\frac{\Psi_m(v_m)}{\sqrt{v_m}})] + (v_m - m)f(\frac{\Psi_m(m)}{\sqrt{m}})$$

$$= -(1 + o(1))m \cdot 2\frac{\psi(m)}{\sqrt{m}}\left(\frac{\psi_m(m)}{\sqrt{m}} - \frac{\psi_m(v_m)}{\sqrt{v_m}}\right)$$

$$+ (1 + o(1)) \frac{m}{\psi^{2r}(m)} \frac{\psi^2(m)}{m}$$

$$= -(1 + o(1)) \cdot 2\sqrt{m}\psi(m)[(\epsilon_m + \frac{\alpha_m}{m}) - (\epsilon_m + \frac{\alpha_m}{v_m})]$$

$$+ (1 + o(1))\psi^{2(1 - r)}(m)$$

$$= -(1 + o(1)) \cdot 2\sqrt{m}\psi(m)\frac{\alpha_m}{m}(1 - \frac{m}{v_m}) + (1 + o(1))\psi^{2(1 - r)}(m)$$

$$= -(1 + o(1)) \cdot 2\sqrt{m}\psi^{1 - 2r}(m) \cdot \frac{\alpha_m}{m} + (1 + o(1))\psi^{2(1 - r)}(m)$$

$$= -(1 + o(1))\sqrt{m}\psi^{1 - 2r}(m) \cdot 2\frac{\alpha_m}{m}$$

$$+ (1 + o(1))\sqrt{m}\psi^{1 - 2r}(m) \cdot (\epsilon_m + \frac{\alpha_m}{m})$$

$$= (1 + o(1))\sqrt{m}\psi^{1 - 2r}(m) \cdot (\epsilon_m - \frac{\alpha_m}{m})$$

$$= (1 + o(1)) \cdot 2m\psi'(m)\psi^{1 - 2r}(m) \quad \text{(recall (2.4.11))}$$

$$\geq (1 + o(1)) \cdot 2\delta\psi^{2(1 - r)}(m) \quad \text{(Lemma 2.2.1(d))}.$$

So

$$\frac{p_{v_m}(\ell_m)}{p_m(\ell_m)} = (1 + o(1)) \frac{P\{S_{v_m} > \ell_m(v_m)\}}{P\{S_m > g(m)\}}$$

$$= (1 + o(1)) \cdot \exp\{-\tfrac{1}{2}[\tilde{\Psi}_m^2(v_m) - \tilde{\Psi}_m^2(m)]\} = o(1),$$

completing the proof of (2.4.13). □

2.5. Line-crossing probabilities.

This section is devoted to the

PROOF of Lemma 2.4.1. We first need to establish some notation. Define the distribution function F_m by

(2.5.1) $\quad F_m(x) = F(\epsilon_m + x);$

in other words, if X is distributed according to F_m, then $X - \epsilon_m$ has distribution function F. The effect of F_m will be to replace the linear boundary ℓ_m by a horizontal one: in obvious notation,

$$p_m(\ell_m) = P_{F_m}\{S_n > \alpha_m \text{ for some } n \geq m\}.$$

Let K_m denote the cumulant generating function corresponding to F_m, so that

(2.5.2) $\quad K_m(\xi) = K(\xi) - \epsilon_m \xi.$

Recall the assumption that K is finite in an open interval I containing 0; K_m is also finite in this interval.

Since F has zero mean and unit variance, it is well-known that

(2.5.3) $\quad K(\xi) \sim \frac{1}{2}\xi^2, \quad K'(\xi) \sim \xi, \quad K''(\xi) \to 1 \quad \text{as} \quad \xi \to 0,$

and that

(2.5.4) $\quad K(0) = 0, \quad K'(0) = 0, \quad K''(\xi) > 0 \quad \text{for all} \quad \xi \in I.$

From (2.5.2) and (2.5.4)

(2.5.5) $\quad K_m(0) = 0, \quad K_m'(0) = -\epsilon_m, \quad K_m''(\xi) > 0 \quad \text{for all} \quad \xi \in I.$

Hence there exists at most one nonzero value $\xi_1(m)$, necessarily positive, for which $K_m(\xi_1(m)) = 0$. We now show that $\xi_1(m)$ exists for sufficiently large m and that

(2.5.6) $\quad \xi_1(m) \sim 2\epsilon_m \quad \text{as} \quad m \to \infty.$

Indeed, let $\xi_m \to 0$ be an arbitrary real sequence. By (2.5.2-3),

(2.5.7) $\quad K_m(\xi_m) = (1 + o(1)) \cdot \frac{1}{2}\xi_m^{\;2} - \epsilon_m \xi_m \quad \text{as} \quad m \to \infty.$

In particular, with $\xi_m \equiv C \cdot 2\epsilon_m$ $(C > 0)$

(2.5.8) $\quad K_m(\xi_m) = 2C\epsilon_m^{\;2}[(1 + o(1))C - 1].$

If $C < 1$, then $K_m(\xi_m) \sim -2C(1 - C)\epsilon_m^{\;2} < 0$; if $C > 1$, then $K_m(\xi_m) \sim 2C(C - 1)\epsilon_m^{\;2} > 0$. This shows that $\xi_1(m)$ exists for sufficiently large m and that, for any pair of constants $0 < C < 1 < C'$, $C < \xi_1(m)/(2\epsilon_m) < C'$ for large m; (2.5.6) follows.

Similarly, for large m there exists exactly one point $\xi_0(m) \in (0, \xi_1(m))$ at which K_m' vanishes, and

(2.5.9) $\quad \xi_0(m) \sim \epsilon_m$ as $m \to \infty$.

This follows from the fact that if $\xi_m \to 0$, then

(2.5.10) $\quad K_m'(\xi_m) = (1 + o(1))\xi_m - \epsilon_m$ as $m \to \infty$.

Next, let P_m denote the probability under which X, X_1, X_2, ... are i.i.d. with

(2.5.11) $\quad P_m\{X \in dx\} = \exp[\xi_0(m) \cdot x - K_m(\xi_0(m))]\, F_m(dx)$.

P_m has cgf

$$\phi_m(\theta) = K_m(\xi_0(m) + \theta) - K_m(\xi_0(m))$$

(2.5.12) $$= K(\xi_0(m) + \theta) - K(\xi_0(m)) - \epsilon_m\theta;$$

note that

(2.5.13) $\quad \phi_m(0) = \phi_m'(0) = 0, \quad \phi_m''(0) = K''(\xi_0(m))$.

Put

(2.5.14) $\quad \theta_0(m) = -\xi_0(m), \quad \theta_1(m) = \xi_1(m) - \xi_0(m)$.

Then

(2.5.15) $\quad \theta_0(m) \sim -\epsilon_m, \quad \theta_1(m) \sim \epsilon_m \quad$ as $\quad m \to \infty$

and

$$\phi_m(\theta_0(m)) = \phi_m(\theta_1(m)) = -K_m(\xi_0(m)) = -(K(\xi_0(m)) + \epsilon_m\theta_0(m))$$

(2.5.16) $$\sim \tfrac{1}{2}\epsilon_m^2 \quad \text{as} \quad m \to \infty .$$

We now introduce the distributions associated through "exponential tilting" with that of X under P_m. For each real θ for which $\phi_m(\theta) < \infty$ let $P_{m,\theta}$ be the probability under which X, $X_1, X_2, \ldots$ are i.i.d. with

$$P_{m,\theta}\{X \in dx\} = \exp[\theta x - \phi_m(\theta)]\, P_m\{X \in dx\}$$

(2.5.17) $$= \exp[\theta x - \phi_m(\theta)]\, \exp[\xi_0(m) \cdot x - K_m(\xi_0(m))]\, F_m(dx).$$

The corresponding cgf $\phi_{m,\theta}$ is given by

$$\phi_{m,\theta}(\eta) = \phi_m(\theta + \eta) - \phi_m(\theta)$$

$$= K_m(\xi_0(m) + \theta + \eta) - K_m(\xi_0(m) + \theta)$$

(2.5.18) $$= K(\xi_0(m) + \theta + \eta) - K(\xi_0(m) + \theta) - \epsilon_m\eta.$$

In particular,

(2.5.19) $\quad E_{m,\theta}X = \phi_{m,\theta}'(0) = \phi_m'(\theta) <, =, \text{ or } > 0$

according as $\theta <, =, \text{ or } > 0$

by (2.5.13) and the strict convexity of ϕ_m.

As special cases of (2.5.17),

$$P_{m,0} = P_m,$$

$$P_{m,\theta_1(m)}\{X \in dx\} = \exp[\xi_1(m) \cdot x]\ F_m(dx),$$

and

$$P_{m,\theta_0(m)}\{X \in dx\}$$

$$= \exp\{[\xi_0(m) + \theta_0(m)] \cdot x - [\phi_m(\theta_0(m)) + K_m(\xi_0(m))]\}\ F_m(dx)$$

$$= F_m(dx) = F(\epsilon_m + dx) \tag{2.5.20}$$

(cf. (2.5.14), (2.5.16), and (2.5.1)). Hence by (2.4.3) and (2.4.5)

$$p_m(\ell_m) = P_{m,\theta_0(m)}\{S_n > \alpha_m \text{ for some } n \geq m\}. \tag{2.5.21}$$

Without loss of generality we consider $P_{m,\theta}$ to be the distribution of $X_1, X_2, \ldots$, defined on the space of (infinite) sequences of real numbers. Accordingly, let $P^{(n)}_{m,\theta}$ denote the restriction of $P_{m,\theta}$ to the σ-algebra generated by the first n coordinates ($n = 1, 2, \ldots$). Then for any θ' and θ'', $P^{(n)}_{m,\theta'}$ and $P^{(n)}_{m,\theta''}$ are mutually absolutely continuous, and by (2.5.17)

$$\frac{dP^{(n)}_{m,\theta'}}{dP^{(n)}_{m,\theta''}} = \exp\{(\theta' - \theta'')S_n - n[\phi_m(\theta') - \phi_m(\theta'')]\}. \tag{2.5.22}$$

In particular, by (2.5.14), (2.5.16), and (2.5.22),

$$\text{(2.5.23)} \qquad \frac{dP^{(n)}_{m,\theta_0(m)}}{dP^{(n)}_{m,\theta_1(m)}} = \exp[-\xi_1(m)S_n] \quad (n = 1, 2, \ldots).$$

Let

$$T_m = \inf\{n: n \geq m, \quad S_n > \alpha_m\},$$

the inf of the empty set being $+\infty$. Then by (2.5.21)

$$\text{(2.5.24)} \qquad p_m(\ell_m) = P_{m,\theta_0(m)}\{S_m > \alpha_m\} + P_{m,\theta_0(m)}\{m < T_m < \infty\}.$$

By (2.4.5) and (2.5.20), the first probability in (2.5.24) is $P\{S_m > \ell_m(m)\}$. To complete the proof of (2.4.9) we shall use the approach of Siegmund (1975) to show that

$$\text{(2.5.25)} \qquad P_{m,\theta_0(m)}\{m < T_m < \infty\} \sim \frac{\epsilon_m + \frac{\alpha_m}{m}}{\epsilon_m - \frac{\alpha_m}{m}} \cdot P\{S_m > \ell_m(m)\} \quad \text{as} \quad m \to \infty .$$

We first apply the fundamental identity of sequential analysis, to wit: by (2.5.23)

$$P_{m,\theta_0(m)}\{m < T_m < \infty\}$$

$$= \sum_{n=m+1}^{\infty} \int_{\{T_m = n\}} \exp[-\xi_1(m)S_n] \, dP_{m,\theta_1(m)}$$

$$= \int_{\{m < T_m < \infty\}} \exp[-\xi_1(m) S_{T_m}] \, dP_{m,\theta_1(m)}$$

$$\text{(2.5.26)} \qquad = \exp[-\xi_1(m)\alpha_m]$$

$$\cdot \int_{\{m < T_m < \infty\}} \exp[-\xi_1(m)(S_{T_m} - \alpha_m)] \, dP_{m,\theta_1(m)}.$$

Recalling $\theta_1(m) > 0$, (2.5.19) implies $E_{m,\theta_1(m)} X > 0$; hence by the SLLN

$$\{m < T_m < \infty\} = \{m < T_m \leq \infty\} = \{S_m \leq \alpha_m\} \quad \text{a.s.} \quad P_{m,\theta_1(m)},$$

and the final integral in (2.5.26) equals

$$\int_{\{S_m \leq \alpha_m\}} \exp[-\xi_1(m)(S_{T_m} - \alpha_m)] \, dP_{m,\theta_1(m)}$$

$$= P_{m,\theta_1(m)}\{S_m \leq \alpha_m\} \cdot E_{m,\theta_1(m)}(\exp[-\xi_1(m)(S_{T_m} - \alpha_m)] \mid S_m \leq \alpha_m)$$

$$\text{(2.5.27)} \qquad = P_{m,\theta_1(m)}\{S_m \leq \alpha_m\}$$

$$\cdot \int_{[0,\infty)} E_{m,\theta_1(m)}(\exp[-\xi_1(m)(S_{T_m} - \alpha_m)] \mid S_m = \alpha_m - y)$$

$$\cdot P_{m,\theta_1(m)}\{S_m \in \alpha_m - dy \mid S_m \leq \alpha_m\}.$$

The last conditional expectation in (2.5.27) is

$$\text{(2.5.28)} \qquad E_{m,\theta_1(m)}(\exp[-\xi_1(m)(S_{\tau(y)} - y)]),$$

where $\tau(y) = \inf\{n: S_n > y\}$. We show below that the expression (2.5.28) tends uniformly in $0 \le y < \infty$ to 1 as $m \to \infty$; thus

$$(2.5.29) \qquad P_{m,\theta_0(m)}\{m < T_m < \infty\} \sim \exp[-\xi_1(m)\alpha_m] \cdot P_{m,\theta_1(m)}\{S_m \le \alpha_m\}.$$

Moreover, mimicking the proof of Cramér's theorem we'll show in Section 2.6 that

$$(2.5.30) \qquad P_{m,\theta_1(m)}\{S_m \le \alpha_m\} \sim \exp[\xi_1(m)\alpha_m] \cdot \frac{\epsilon_m + \frac{\alpha_m}{m}}{\epsilon_m - \frac{\alpha_m}{m}} \cdot P\{S_m > \ell_m(m)\}.$$

Then (2.5.25) follows from (2.5.29) and (2.5.30).

The first of our two assertions is that

$$(2.5.31) \qquad E_{m,\theta_1(m)}(\exp[-\xi_1(m)R_y]) \to 1 \quad \text{as} \quad m \to \infty,$$

$$\text{uniformly in} \quad 0 \le y < \infty,$$

where $R_y = S_{\tau(y)} - y$ is the excess over the horizontal boundary with level y. In light of the inequality $1 \ge e^{-\eta} \ge 1 - \eta$ for $\eta \ge 0$ and (2.5.16) it is enough to show that the nonnegative quantity

$$(2.5.32) \qquad \sup\{E_{m,\theta_1(m)}R_y : 0 \le y < \infty\} = o(\tfrac{1}{\epsilon_m}) \quad \text{as} \quad m \to \infty.$$

Indeed, for $0 \le y < \infty$ we find

$$(E_{m,\theta_1(m)}R_y)^2 \le E_{m,\theta_1(m)}R_y^2$$

$$\le \frac{4}{3} E_{m,\theta_1(m)}(X^+)^3/E_{m,\theta_1(m)}X$$

using theorem 3 in Lorden (1970). An easy dominated convergence argument shows that $E_{m,\theta_1(m)}(X^+)^3 \rightarrow E(X^+)^3 < \infty$, and by (2.5.19),(2.5.12),(2.5.14),(2.5.10), and (2.5.6)

$$E_{m,\theta_1(m)}X = \phi_m'(\theta_1(m)) = K'(\xi_1(m)) - \epsilon_m$$

$$= (1 + o(1))\xi_1(m) - \epsilon_m = (1 + o(1))\epsilon_m. \tag{2.5.33}$$

Therefore

$$\sup\{E_{m,\theta_1(m)}R_y\colon 0 \le y < \infty\} \le (1 + o(1))[\tfrac{4}{3} E(X^+)^3/\epsilon_m]^{1/2}$$

$$= O(\epsilon_m^{-1/2}) = o(\epsilon_m^{-1}). \tag{2.5.34}$$

which proves (2.5.32).

The remaining assertion is (2.5.30). Standardizing to zero mean and unit variance,

$$P_{m,\theta_1(m)}\{S_m \le \alpha_m\} = P_{m,\theta_1(m)}\{\frac{S_m - mE_{m,\theta_1(m)}X}{(mVar_{m,\theta_1(m)}X)^{1/2}} \le -u_m\}$$

with

$$u_m = -(\alpha_m - mE_{m,\theta_1(m)}X)/(mVar_{m,\theta_1(m)}X)^{1/2}. \tag{2.5.35}$$

From (2.5.33) and

$$(2.5.36)\qquad \mathrm{Var}_{m,\theta_1(m)}X = \phi_m''(\theta_1(m)) = K_m''(\xi_1(m)) = K''(\xi_1(m)) \to 1$$

and (2.4.8) follows

$$(2.5.37)\qquad u_m \sim \sqrt{m}(\sigma_m - \frac{\alpha_m}{m}).$$

Let G_m be the standardized distribution of $(-X)$ under $P_{m,\theta_1(m)}$, and let λ_m be the associated Cramér series.
Were it not for the dependence of G_m on m, direct application of Cramér's result (which holds as well when $>$ is changed to $\geq$ on the left in (1.2.7)) would give

$$(2.5.38)\qquad P_{m,\theta_1(m)}\{S_m \leq \alpha_m\} \sim P\{Z > u_m\}\exp[\frac{u_m^3}{\sqrt{m}}\lambda_m(\frac{u_m}{\sqrt{m}})].$$

In fact, (2.5.38) can be established by rehashing the proof of Cramér's theorem and used to deduce (2.5.30). We shall follow a somewhat shorter route and verify (2.5.30) directly, but our proof, like Cramér's, will be based on the standard large-deviation technique of exponential tilting.

2.6. A Cramér-like result.

In this section we complete the proof of Lemma 2.4.1 by establishing the Cramér-like result (2.5.30).

For abbreviation we introduce the notation

$$(2.6.1)\qquad \Psi(m) = \ell_m(m)/\sqrt{m} = \sqrt{m}(\epsilon_m + \frac{\alpha_m}{m}).$$

This should cause no confusion; after all, whenever Lemma 2.4.1 is applied to the original convergence rate problem the identification (2.6.1) is made. In addition, let

$$(2.6.2)\qquad z_m = z(\frac{\Psi(m)}{\sqrt{m}})$$

with z defined in (2.1.1) and put

$$(2.6.3)\qquad \theta_2(m) = z_m + \theta_0(m);$$

we shall soon tilt from $P_{m,\theta_1(m)}$ to $P_{m,\theta_2(m)}$ to compute the left side of (2.5.30). *Observe*

$$(2.6.4)\qquad z_m \sim \frac{\Psi(m)}{\sqrt{m}} \to 0,$$

$$(2.6.5)\qquad \theta_2(m) = \frac{\alpha_m}{m} + o(\epsilon_m) \to 0,$$

$$(2.6.6)\qquad \phi_m(\theta_2(m)) = K(z_m) - K(\zeta_0(m)) - \epsilon_m\theta_2(m).$$

Also note that

$$(2.6.7)\qquad E_{m,\theta_2(m)}X = \phi_m'(\theta_2(m)) = K'(z_m) - \epsilon_m = \frac{\Psi(m)}{\sqrt{m}} - \epsilon_m = \frac{\alpha_m}{m}$$

and that

$$(2.6.8)\qquad \sigma_m^{\ 2} \equiv \mathrm{Var}_{m,\theta_2(m)}X = \phi_m''(\theta_2(m)) = K''(z_m) \to 1.$$

We are going to use the Berry-Esséen theorem, and so the third absolute moments

$$(2.6.9) \qquad \rho_m = E_{m,\theta_2(m)}|X|^3, \quad \rho = E|X|^3 < \infty$$

will arise; dominated convergence gives

$$(2.6.10) \qquad \rho_m \to \rho.$$

Let

$$(2.6.11) \qquad \pi_m = P_{m,\theta_1(m)}\{S_m \leq \alpha_m\}$$

denote the left side of (2.5.30). Putting $\theta' = \theta_1(m)$, $\theta'' = \theta_2(m)$, and $n = m$ in (2.5.22) yields

$$(2.6.12) \qquad \pi_m = \exp\{-m[\phi_m(\theta_1(m)) - \phi_m(\theta_2(m))]\} \cdot \int_{(-\infty,\alpha_m]} \exp[(\theta_1(m) - \theta_2(m))s]\, P_{m,\theta_2(m)}\{S_m \in ds\}.$$

Recalling (2.5.16) and (2.6.6) and simplifying,

$$(2.6.13) \qquad \pi_m = \exp\{-m[\epsilon_m z_m - K(z_m)]\} \cdot \int_{(-\infty,\alpha_m]} \exp[(\theta_1(m) - \theta_0(m) - z_m)s] \cdot P_{m,\theta_2(m)}\{S_m \in ds\}.$$

If the approximation $\hat{\pi}_m$ to π_m is obtained from the right

side of (2.6.13) by replacing $P_{m,\theta_2(m)}\{S_m \in ds\}$ with the normal distribution $P\{\alpha_m + m^{1/2}\sigma_m Z \in ds\}$ with the same mean and variance, then (completing the square and rearranging)

$$\pi_m = \exp[(\theta_1(m) - \theta_0(m))\alpha_m]$$

$$\cdot \exp[\tfrac{1}{2}m\sigma_m{}^2(\theta_1(m) - \theta_2(m))^2]\, P\{Z \le -\sqrt{m}\sigma_m(\theta_1(m) - \theta_2(m))\}$$

$$\cdot \exp\{m[K(z_m) - z_m \cdot \frac{\Psi(m)}{\sqrt{m}}]\}$$

(2.6.14) $$= \exp[(\theta_1(m) - \theta_0(m))\alpha_m]$$

$$\cdot \exp[-\tfrac{1}{2}\Psi^2(m) + h(\sqrt{m}\sigma_m(\theta_1(m) - \theta_2(m)))]$$

$$\cdot \exp[\frac{\Psi^3(m)}{\sqrt{m}}\lambda(\frac{\Psi(m)}{\sqrt{m}})],$$

where

(2.6.15) $$h(t) = \log(e^{\frac{1}{2}t^2} \cdot P\{Z > t\}).$$

Note

$$h'(t) = t - \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}/P\{Z > t\}$$

(2.6.16) $$\sim -1/t \quad \text{as} \quad t \to \infty .$$

So by the mean value theorem

$$h(\sqrt{m}\sigma_m(\theta_1(m) - \theta_2(m))) - h(\sqrt{m}(\epsilon_m - \frac{\alpha_m}{m}))$$

$$= -(1 + o(1))\sqrt{m}[\sigma_m(\theta_1(m) - \theta_2(m)) - (\epsilon_m - \frac{\alpha_m}{m})]/[\sqrt{m}(\epsilon_m - \frac{\alpha_m}{m})]$$

(2.6.17) $\quad = o(1),$

and hence the second of the three factors on the right side of (2.6.14) is $(1 + o(1))$ times

$$\exp[-\frac{1}{2}\Psi^2(m) + h(\sqrt{m}(\epsilon_m - \frac{\alpha_m}{m}))]$$

$$= \exp[-\frac{1}{2}\Psi^2(m)]\exp[\frac{1}{2}m(\epsilon_m - \frac{\alpha_m}{m})^2]P\{Z > \sqrt{m}(\epsilon_m - \frac{\alpha_m}{m})\}$$

$$= (1 + o(1))(\epsilon_m - \frac{\alpha_m}{m})^{-1}(2\pi m)^{-1/2}\exp[-\frac{1}{2}\Psi^2(m)]$$

$$= (1 + o(1))\,\frac{\epsilon_m + \frac{\alpha_m}{m}}{\epsilon_m - \frac{\alpha_m}{m}}\,[\sqrt{2\pi}\Psi(m)]^{-1}\,\exp[-\frac{1}{2}\Psi^2(m)]$$

$$= (1 + o(1))\,\frac{\epsilon_m + \frac{\alpha_m}{m}}{\epsilon_m - \frac{\alpha_m}{m}}\,P\{Z > \Psi(m)\}.$$

In light of (1.2.7) it is now clear that

(2.6.18) $\quad \pi_m \sim$ right side of (2.5.30).

Moreover, integration by parts in (2.6.13) leads to the error estimate

$$(2.6.19) \qquad |\pi_m - \hat{\pi}_m| \leq 2C \cdot \frac{\rho_m}{\sigma_m{}^3\sqrt{m}} \cdot \exp[(\theta_1(m) - \theta_0(m))\alpha_m]$$

$$\cdot \exp\{m[K(z_m) - z_m \cdot \frac{\Psi(m)}{\sqrt{m}}]\}$$

where C is the universal constant appearing in the Berry-Eséen bound $C \cdot \rho/(\sigma^3 n^{1/2})$ on the error in the central limit theorem (see Feller, 1971, p. 542). Comparing with (2.6.14) and recalling (2.6.15), the right side of (2.6.19) is

$$(2.6.20) \qquad 2C \cdot \frac{\rho_m}{\sigma_m{}^3\sqrt{m}} \cdot \hat{\pi}_m \exp[-h(\sqrt{m}\sigma_m(\theta_1(m) - \theta_2(m)))].$$

But

$$(2.6.21) \qquad \exp[-h(t)] \sim \sqrt{2\pi}t \quad \text{as} \quad t \rightarrow \infty,$$

so by (2.6.8), (2.6.10), and (2.6.21), (2.6.20) is

$$(1 + o(1)) \cdot 2C\rho\sqrt{2\pi}(\epsilon_m - \frac{\alpha_m}{m})\hat{\pi}_m = o(\hat{\pi}_m).$$

Together $\pi_m \sim \hat{\pi}_m$ and (2.6.18) give (2.5.30), completing the proof of Lemma 2.4.1. $\square$

CHAPTER III

STRASSEN AND SIEGMUND REVISITED

3.1. Introduction.

The following theorem on boundary crossing probabilities for Brownian motion B is due to Strassen (1965, cf. thm. 1.2).

As in Chapter 2, let $g: (0, \infty) \to (0, \infty)$ and write

$$g(t) = \sqrt{t}\,\psi(t)$$

THEOREM 3.1.1 (Strassen). Suppose that g is continuously differentiable with

$$(3.1.1) \qquad \frac{g(t)}{t^{\zeta}} \uparrow$$

as $t \uparrow$ for some $\zeta > 0$. Assume as in (2.1.8) that

$$(3.1.2) \qquad g'(u) \sim g'(t) \quad \text{as} \quad u \sim t \to \infty$$

and that $g \in U(B)$. Then

$$(3.1.3) \qquad P\{B(t) > g(t) \text{ for some } t \geq s\} \sim J_s \quad \text{as} \quad s \to \infty,$$

where

$$(3.1.4) \qquad J_s \equiv \int_s^\infty \frac{1}{\sqrt{2\pi}} \frac{g'(t)}{\sqrt{t}} e^{-\frac{1}{2}\psi^2(t)} dt. \quad \square$$

Approximating the random walk S by B using Skorohod embedding, Strassen deduced Theorem 1.4.1. For Skorohod embedding Strassen was able to show that

$$(3.1.5) \qquad S_n = B(n) + O(n^{1/4}(\log n)^{1/2}(\log \log n)^{1/4}) \text{ a.s. as } n \to \infty,$$

but this bound is crude enough that Strassen was forced to impose the restriction $g(t) \leq t^{3/5 - \gamma}$ in place of the more natural (cf. (1.2.9a)) $g(t) = o(t^{2/3})$. Furthermore, as explained in Remark 1.4.2(b), Strassen's use of the Skorohod technique is flawed.

A response to both criticisms is provided by the approximation scheme of Komlós, Major, and Tusnády (1975; 1976). These authors used techniques strikingly different from those of Skorohod to obtain a better approximation of B to S. The resulting improvement (Komlós, Major, and Tusnády, 1976, thm. 1) to (3.1.5) is

$$(3.1.6) \qquad S_n = B(n) + O(\log n) \text{ a.s. as } n \to \infty,$$

and Komlós et al. showed that (3.1.6) is the best possible result in this direction. In Theorem 3.2.1 we use Theorem 3.1.1 and the Komlós et al. approximation to give a correct proof of Theorem 1.4.1, widening the range of boundaries to $g(t) = o(t^{2/3})$. Corollary 3.2.3 provides a neat summary of Theorems 2.1.1 and 3.2.1.

In Section 3.4 we generalize Theorem 1.5.1 to linear boundaries g with nonzero intercept.

3.2. Strassen revisited.

THEOREM 3.2.1. Adopt all the assumptions of Section 2.1 preceding (2.1.6), relaxing (2.1.3) to

$$\frac{g(t)}{t^{1/2}} \uparrow \tag{3.2.1}$$

and tightening (2.1.4) to

$$\frac{g(t)}{t^{2/3}} \downarrow 0 \tag{3.2.2}$$

and (2.1.5) to (3.1.2). If $g \in U$, then

$$p_m \sim J_m \quad \text{as} \quad m \to \infty \tag{3.2.3}$$

with $J_\cdot$ given by (3.1.4). ▯

REMARK 3.2.2. (a) As with Theorems 2.1.1 and 3.1.1, the various regularity conditions imposed on g need only hold for large t.

(b) We remark without proof that condition (3.1.2) can be relaxed to (2.1.5). One must then assume, however, that

$$\frac{g(t)}{(tL_\rho t)^{1/2}} \uparrow \tag{3.2.4}$$

for some ρ or, more generally, that

$$(3.2.5) \qquad \frac{g(t)}{(tf(t))^{1/2}} \uparrow$$

for some function f satisfying

$$(3.2.6) \qquad f(t) \uparrow \infty, \quad (\log f)'(t) \geq t^{-5/4}$$

for large t. $\square$

We postpone the proof of Theorem 3.2.1 to remark that Lemma 3.2.4 below allows us to combine Theorems 2.1.1 and 3.2.1 to form a strong-law analogue to Cramér's theorem:

COROLLARY 3.2.3. Adopt all the assumptions of Section 2.1 preceding (2.1.6), easing the restriction on δ to $0 \leq \delta < 1/2$ and tightening (2.1.4) to

$$(3.2.7) \qquad \frac{g(t)}{t^{1-\eta}} \downarrow 0$$

for some $0 \leq \eta < 1/2$ and (2.1.5) to (3.1.2). If $g \in U$ and either $\delta > 0$ or $\eta \geq 1/3$, then

$$(3.2.8) \qquad p_m \sim \tilde{J}_m \equiv \int_m^\infty \frac{1}{\sqrt{2\pi}} \frac{g'(t)}{\sqrt{t}} e^{-\frac{1}{2}\Psi^2(t)} \exp[\Psi^2(t)\frac{\Psi(t)}{\sqrt{t}}\lambda(\frac{\Psi(t)}{\sqrt{t}})]\, dt. \quad \square$$

Lemma 3.2.4 obtains the alternative integral expression (3.2.8) for the rate of convergence of p_m in the case of Theorem 2.1.1.

<u>LEMMA</u> 3.2.4. Under the assumptions of Theorem 2.1.1, (3.2.8) holds.

<u>PROOF</u>. According to (2.3.6) and the normal tail estimate (1.2.8),

$$P\{S_m > g(m)\} \sim P\{Z > \tilde{\Psi}(m)\}$$

with $\tilde{\Psi}$ defined in (2.3.7). It follows (cf. (2.3.2), (2.3.3), (2.1.6), (2.1.5), and Lemma 2.2.1(j)) that

$$p_{v_m} \sim \frac{g'(m)}{\sqrt{m}\tilde{\Psi}'(m)} P\{Z > \tilde{\Psi}(v_m)\}$$

and hence from Lemma 2.3.1 and (2.3.9) that

$$p_m = (1 + o(1)) \frac{g'(m)}{\sqrt{m}\tilde{\Psi}'(m)} P\{\tilde{\Psi}(m) < Z \le \tilde{\Psi}(v_m)\}$$

$$= (1 + o(1)) \frac{g'(m)}{\sqrt{m}\tilde{\Psi}'(m)} \int_m^{v_m} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\tilde{\Psi}^2(t)} \tilde{\Psi}'(t)\, dt$$

$$= (1 + o(1)) \int_m^{v_m} \frac{1}{\sqrt{2\pi}} \frac{g'(t)}{\sqrt{t}} e^{-\frac{1}{2}\tilde{\Psi}^2(t)}\, dt$$

$$= (1 + o(1))(\tilde{J}_m - \tilde{J}_{v_m}). \tag{3.2.9}$$

But using Lemma 2.2.1(k) we find

$$\tilde{J}_m = \int_m^{\infty} \frac{1}{\sqrt{2\pi}} \frac{g'(t)}{\sqrt{t}} e^{-\frac{1}{2}\tilde{\Psi}^2(t)}\, dt$$

$$\le (1 + \frac{1}{2\delta}) \int_m^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\tilde{\Psi}^2(t)} \tilde{\Psi}'(t)\, dt$$

$$= (1 + o(1))\ (1 + \frac{1}{2\delta}) \int_m^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\tilde{\Psi}^2(t)} \tilde{\Psi}'(t)\, dt$$

$$= (1 + o(1))\ (1 + \frac{1}{2\delta})\ P\{Z > \tilde{\Psi}(m)\}$$

$$= (1 + o(1))\ (1 + \frac{1}{2\delta})\ P\{S_m > g(m)\}$$

and similarly

$$\tilde{J}_m \geq (1 + o(1))2P\{S_m > g(m)\}.$$

Hence

$$(1 + o(1))\ \frac{2}{1 + \frac{1}{2\delta}} \leq \tilde{J}_m/p_m \leq (1 + o(1))\ \frac{1}{2}\ (1 + \frac{1}{2\delta}).$$

Thus

$$(3.2.10) \qquad 0 \leq \tilde{J}_{v_m} \leq (1 + o(1))\ \frac{1}{2}\ (1 + \frac{1}{2\delta})\ p_{v_m} = o(p_m) = o(\tilde{J}_m)$$

and (3.2.8) follows from (3.2.9-10). $\square$

We now undertake the proof of Theorem 3.2.1. Using the approximation of Komlós et al. we shall prove in Section 3.3 the following strong invariance principle to be used in conjunction with Theorem 3.1.1.

Extend the time domain of S to $[0, \infty)$ via the definition

$$(3.2.11) \qquad S(t) = S_{\lfloor t \rfloor}, \quad 0 \leq t < \infty.$$

LEMMA 3.2.5. Adopt the assumptions of Theorem 3.2.1. Choose $\alpha > 0$ and put

$$(3.2.12) \qquad h = g^{\alpha}.$$

Then, without loss of generality, there exist a Brownian motion B and positive constants K and M such that

$$(3.2.13) \qquad P\{|S(t) - B(t)| > h(t) \text{ for some } t \geq s\} \leq Ke^{-Mh(s)}$$

for all $s > 0$. □

REMARK 3.2.6. (a) The phrase "without loss of generality" is to be interpreted in the specific sense of Strassen (1965): there exists a probability space on which S and B are both defined and (3.2.13) holds. The proof of Theorem 3.2.1 then applies to this version of S, but $p_m = P\{S_n > g(n) \text{ for some } n \geq m\}$ is clearly the same for any version of S.

(b) The constants K and M depend on F, g, and α. □

In proving Theorem 3.2.1 we apply Lemma 3.2.5 with $\alpha = 1/2$. First observe that

$$p_m = P\{S(t) > g(t) \text{ for some } t \geq m\}.$$

Hence

$$P\{B(t) > g(t) + h(t) \text{ for some } t \geq m\} - Ke^{-Mh(m)}$$

$$\leq p_m \leq P\{B(t) > g(t) - h(t) \text{ for some } t \geq m\} + Ke^{-Mh(m)}$$

with $h = g^{1/2}$. Now

$$p_m \geq P\{S_m > g(m)\} = (1 + o(1))[\sqrt{2\pi}\tilde{\psi}(m)]^{-1}\exp[-\tfrac{1}{2}\tilde{\psi}^2(m)]$$

$$= \exp[-(1 + o(1))\tfrac{1}{2}\psi^2(m)]$$

and by (3.2.2)

$$\psi^2(m) = o(h(m)),$$

so

$$Ke^{-Mh(m)} = o(p_m)$$

and

$$(1 + o(1))P\{B(t) > g(t) + h(t) \text{ for some } t \geq m\}$$

(3.2.14) $$\leq p_m \leq (1 + o(1))P\{B(t) > g(t) - h(t) \text{ for some } t \geq m\}.$$

To complete the proof of Theorem 3.2.1 we'll show that both of the Brownian motion probabilities in (3.2.14) equal $(1 + o(1))\, J_m$.

Begin with the right side of (3.2.14). Because $g \in U$ and $\psi\uparrow$, the KPEF test (see Section 1.3) implies that

$$\int^{\infty} \frac{\psi(t)}{t}\, e^{-\frac{1}{2}\psi^2(t)}\, dt < \infty .$$

Since $g - h$ satisfies hypothesis (3.1.1) of Theorem 3.1.1 with $\zeta = 1/2$ and by (3.2.2)

$$\Psi(t)\frac{h(t)}{\sqrt{t}} = \frac{g^{3/2}(t)}{t} = o(1),$$

it follows from the KPEF test that $g - h \in U$. Moreover, $g - h$ also satisfies (3.1.2) because

$$h'(t) = \frac{g'(t)}{2h(t)} = o(g'(t)).$$

Thus (3.1.3) holds for $g - h$:

$$P\{B(t) > g(t) - h(t) \text{ for some } t \geq m\}$$

$$= (1 + o(1)) \int_m^\infty \frac{1}{\sqrt{2\pi}} \frac{g'(t) - h'(t)}{\sqrt{t}} \exp\{-\tfrac{1}{2}(\Psi(t) - \frac{h(t)}{\sqrt{t}})^2\}\, dt$$

$$= (1 + o(1))J_m \text{ as } m \to \infty. \tag{3.2.15}$$

As for the left side of (3.2.14), we note that $g \in U$ and $g + h \geq g$ implies $g + h \in U$, whether U equals $U(S)$ or $U(B)$. The function $g + h$ also satisfies the assumptions of Theorem 3.1.1, but now we need to take $\zeta = 1/4$. Proceeding just as with the right side, we find the left side of (3.2.14) to be $(1 + o(1)) J_m$, and the proof is complete. □

3.3. A strong invariance principle.

Using the standard probability estimate

$$P\{\sup_{n \leq t \leq n+1} |B(t) - B(n)| > x\} = P\{\sup_{0 \leq t \leq 1} |B(t)| > x\}$$

$$\leq 4P\{B(1) > x\} \tag{3.3.1}$$

for the uniformity of B over an interval $[n, n + 1]$ of length one,

one can prove without difficulty the following slight extension of theorem 1 in Komlós, Major, and Tusnády (1976).

LEMMA 3.3.1. Let S satisfy the assumptions of Section 2.1, including restriction to the mgf case. Then, without loss of generality, there exist a Brownian motion B and positive constants a, b, c depending only on F such that

$$(3.3.2) \qquad P\{\sup_{0\le t\le s} |S(t) - B(t)| > x\} \le as^b e^{-cx}$$

for all $x > 0$ and $s \ge 1$. □

With this result we are now ready for the

PROOF of Lemma 3.2.5. We may suppose that $s > 1$ and define $\ell \ge 0$ from s by

$$(3.3.3) \qquad 2^{\ell} < s \le 2^{\ell+1}.$$

Then by Lemma 3.3.1 and the increasingness of h

$$\begin{aligned} p(s) &\equiv P\{|S(t) - B(t)| > h(t) \text{ for some } t \ge s\} \\ &\le P(\cup_{j\ge\ell} \{|S(t) - B(t)| > h(t) \text{ for some } 2^j < t \le 2^{j+1}\}) \\ &\le \Sigma_{j\ge\ell} P\{\sup_{0\le t\le 2^{j+1}} |S(t) - B(t)| > h(2^j)\} \\ &\le a_1 \Sigma_{j\ge\ell} \exp[b_1 j - c_1 h(2^j)] \end{aligned}$$

with $a_1 = a \cdot 2^b$, $b_1 = b \cdot \log 2$, and $c_1 = c$. According

to (3.2.1), $h(2^j) \geq h(1) \cdot 2^{\alpha j/2}$ for all $j \geq 0$, and so (3.3.4) implies that for $c_2 = c_1/2$ and some $a_2 > 0$

$$p(s) \leq a_2 \sum_{j \geq \ell} \exp[-c_2 h(2^j)]$$

$$\leq a_2 \int_{\ell-1}^{\infty} \exp[-c_2 h(2^t)]\, dt$$

$$= \frac{a_2}{\log 2} \int_{2^{\ell-1}}^{\infty} u^{-1} \exp[-c_2 h(u)]\, du.$$

Now by (3.2.12) and the fact $g'(u) \geq g(u)/(2u)$ (cf. Lemma 2.2.1(b))

$$h'(u) = \alpha h(u) \frac{g'(u)}{g(u)} \geq \frac{1}{2} \alpha \frac{h(u)}{u};$$

hence with $a_3 = (2a_2)/[\alpha c_2 h(1/2) \log 2] > 0$ we find

$$p(s) \leq a_3 \int_{2^{\ell-1}}^{\infty} \exp[-c_2 h(u)]\, c_2\, h'(u)\, du$$

$$= a_3 \exp[-c_2 h(2^{\ell-1})]$$

$$\leq a_3 \exp[-c_2 h(\tfrac{1}{4}s)].$$

(3.2.13) now follows with $K = a_3$ and $M = 4^{-2\alpha/3} c_2$ from (3.2.2). ▯

3.4. Siegmund revisited.

In solving the convergence rate problem in the SLLN case (see Section 1.5), Siegmund (1975) treated linear boundaries $g(t) = \epsilon t$ that pass through the origin and have positive slope ϵ. Our proof of Theorem 2.1.1 used Siegmund's techniques to establish the key Lemma 2.4.1, which estimates crossing probabilities for boundaries $\ell_m(t) = \alpha_m + \epsilon_m t$ having variable intercept α_m and slope $\epsilon_m = o(1)$. The provability of Lemma 2.4.1 leads to the conjecture that Siegmund's result can be extended to the case $g(t) = \alpha + \epsilon t$. The purpose of this section is to state such an extension, Siegmund's proof adapting easily to our generalization.

Accordingly, let $\epsilon > 0$ and $\alpha \in \mathbb{R}$ be given, and denote by g the straight line

$$(3.4.1) \qquad g(t) = \alpha + \epsilon t.$$

Our goal is to determine the asymptotic rate at which the boundary crossing probabilities

$$(3.4.2) \qquad p_m = P\{S_n > g(n) \text{ for some } n \geq m\}$$

converge to zero as $m \to \infty$ when S satisfies the assumptions of Section 1.1.

We need some notation. Define

$$K_\epsilon(\xi) = K(\xi) - \epsilon\xi$$

for ξ in the interval of finiteness of K. As at the start of the proof of Lemma 2.4.1 it follows that if ϵ is small enough then there exists a unique nonzero value ξ_1, necessarily positive, for which $K_\epsilon(\xi_1) = 0$. Assume that such a value ξ_1 exists -- this is the criterion of "smallness" for ϵ mentioned in Section 1.5 — and let ξ_0 denote the point in $(0, \xi_1)$ at which K_ϵ' vanishes. In the notation of (2.1.1), $\xi_0 = z(\epsilon)$.

As (2.5.17) and (2.5.11) defined $P_{m,0}$, so let P_0 denote the probability under which $X_1, X_2, \ldots$ are i.i.d. with

(3.4.3) $\quad P_0\{X \in dx\} = \exp[\xi_0 x - K_\epsilon(\xi_0)]\ F(\epsilon + dx);$

P_0 has cgf

(3.4.4) $\quad \phi(\theta) = K_\epsilon(\xi_0 + \theta) - K_\epsilon(\xi_0),$

mean

(3.4.5) $\quad \phi'(0) = K_\epsilon'(\xi_0) = 0,$

and variance

(3.4.6) $\quad \sigma^2 \equiv \phi''(0) = K_\epsilon''(\xi_0) = K''(\xi_0).$

Put

(3.4.7) $\quad \theta_0 = -\xi_0 < 0;$

then (cf. (2.1.1))

$$(3.4.8) \qquad \phi(\theta_0) = -K_{\epsilon}(\zeta_0) = \tfrac{1}{2}\epsilon^2 - \epsilon^3\lambda(\epsilon).$$

We further assume that either

(3.4.9) F is non-lattice

or

(3.4.10) $F(\epsilon + \cdot)$ is a lattice distribution supported by $\{0, \pm h, \pm 2h, \ldots\}$ and with span $h > 0$, and $\alpha = hk$ for some (positive, negative, or zero) integer k.

In Siegmund's paper the necessary (see the proof of his lemma 1) assumption that h is the span of $F(\epsilon + \cdot)$ (rather than just a multiple thereof) is tacit. The distinction is not merely academic, for if, for example, F is a lattice distribution supported by the multiples of h and with span h, then (3.4.10) implies $\epsilon \geq h$. For another illustration, in the case of symmetric Bernoulli components ($X = \pm 1$ with probability $1/2$ each), Siegmund's theorem and the present generalization are vacuuous(!), for the smallest possible value $\epsilon = 1$ still does not satisfy the above criterion of smallness.

Generalizing, and simplifying the statement of, theorem 1 in Siegmund (1975) we have

THEOREM 3.4.1. Using the preceding assumptions and notation, as $m \to \infty$

$$(3.4.11)\qquad p_m \sim \exp\{\textstyle\sum_{n=1}^{\infty} n^{-1}\,[e^{-n\phi(\theta_0)}P_0\{S_n > 0\} - P\{S_n > \epsilon n\}]\}$$
$$\cdot\ P\{S_m > g(m)\}$$

and furthermore

$$(3.4.12)\qquad P\{S_m > g(m)\} \sim C(2\pi\sigma^2 m)^{-1/2}\exp(-m\phi(\theta_0) + \theta_0\alpha),$$

where if (3.4.9) holds

$$(3.4.13)\qquad C = 1/|\theta_0|$$

and if (3.4.10) holds

$$(3.4.14)\qquad C = h/[\exp(|\theta_0|h) - 1]. \quad \square$$

Notice that the only factor in (3.4.11-12) depending on α is $\exp(\theta_0\alpha)$.

CHAPTER IV

TWO REMARKS

4.1. Introduction.

In Section 1.4 we stated that Strassen (1965) determined the probabilities

$$p_m = P\{S_n > g(n) \quad \text{for some} \quad n \ge m\}$$

up to a factor $(1 + o(1))$ under certain conditions on g. In fact, Strassen's theorem 1.4 specifies the convergence rate for the probabilities

$$P\{S_n \ge g(n) \quad \text{for some} \quad n > m\}.$$

The results are the same: in Section 4.2 we show that under the assumptions of Corollary 3.2.3 the four probabilities

$$(4.1.1) \qquad p_m = P\{S_n > g(n) \quad \text{for some} \quad n \ge m\},$$

$$(4.1.2) \qquad p_{m+1} = P\{S_n > g(n) \quad \text{for some} \quad n > m\},$$

$$(4.1.3) \qquad p_m^+ = P\{S_n \ge g(n) \quad \text{for some} \quad n \ge m\},$$

$$(4.1.4) \qquad p_{m+1}^+ = P\{S_n \ge g(n) \quad \text{for some} \quad n > m\}$$

are all asymptotically equivalent, i.e., equal up to a factor $(1 + o(1))$, as $m \to \infty$. Section 4.2 also contains the noteworthy result that, under the assumptions of Theorem 3.4.1, p_m and p_{m+1} are not asymptotically equivalent; furthermore, that $p_m^+ \sim p_m$ only in the non-lattice case (3.4.9).

Section 4.3 is concerned with an entirely different matter. Let S satisfy the assumptions of Section 1.1 and let g be a boundary belonging to both $U(S)$ and $U(-S)$. According to the definition of upper class boundaries the probabilities (4.1.1) and

$$\tilde{p}_m \equiv P\{S_n < -g(n) \text{ for some } n \geq m\}$$

both vanish as $m \to \infty$. The double-boundary crossing probability

$$P\{|S_n| > g(n) \text{ for some } n \geq m\} \tag{4.1.5}$$

is majorized by $p_m + \tilde{p}_m$ and so is also small when m is large. One expects that the probability of the event that S crosses one of the boundaries $\pm g$ after time m and then alters its course so drastically as to cross the other is of smaller order of magnitude than (4.1.5). This is indeed the case, as the following theorem, proved in Section 4.3, demonstrates:

THEOREM 4.1.1. Let $b \in U(S)$, $a \in U(-S)$, with $a \uparrow$, $b \uparrow$. Then

$$P\{S_n > b(n) \text{ or } S_n < -a(n) \text{ for some } n \geq m\}$$

$$(4.1.6) \qquad \sim P\{S_n > b(n) \text{ for some } n \geq m\} + P\{S_n < -a(n) \text{ for some } n \geq m\}. \quad \square$$

Section 4.3 also discusses some consequences of Theorem 4.1.1 when $a = b = g$.

4.2. $>$ versus $\geq$.

Suppose, as in the hypotheses of Corollary 3.2.3, that the boundary $g(t) = t^{1/2}\Psi(t)$ satisfies the growth conditions

$$(4.2.1) \qquad \frac{g(t)}{\sqrt{t}} \uparrow \infty, \quad \frac{g(t)}{t} \downarrow 0$$

as $t \uparrow \infty$. Cramér's result (1.2.7) can be stated in the form

$$(4.2.2) \qquad P\{S_m > g(m)\} \sim P\{Z > \tilde{\Psi}(m)\} \quad \text{as } m \to \infty$$

with $\tilde{\Psi}$ defined by (2.3.7). Now as $t \to \infty$, $\tilde{\Psi}(t) \sim \Psi(t) \to \infty$ and, by a minor adjustment to the proof of (2.3.10), $\tilde{\Psi}^2(t+1) - \tilde{\Psi}^2(t) \to 0$. It follows using the normal tail estimate (1.2.8) and Lemma 2.2.1(h) (whose proof requires only (4.2.1)) that as $m \to \infty$

$$(4.2.3) \qquad P\{S_{m+1} > g(m+1)\} \sim P\{S_m > g(m)\}.$$

The strong-law counterpart

$$(4.2.4) \qquad p_{m+1} \sim p_m$$

to (4.2.3) is established using the result

$$(4.2.5) \qquad p_m \sim \tilde{J}_m = \int_m^{\infty} \frac{1}{\sqrt{2\pi}} \frac{g'(t)}{\sqrt{t}} e^{-\frac{1}{2}\tilde{\Psi}^2(t)} dt$$

of Corollary 3.2.3. Indeed it is enough to recall that $\tilde{\Psi}^2(t+1) - \tilde{\Psi}^2(t) \rightarrow 0$ and to observe from (3.1.2) that $g'(t+1) \sim g'(t)$ as $t \rightarrow \infty$.

We next consider boundary crossing probabilities in which the definition of the event { S crosses g at time n } is changed from the strict inequality $\{S_n > g(n)\}$ to $\{S_n \geq g(n)\}$. If the component distribution F is continuous, i.e., has no point masses, then clearly $P\{S_m \geq g(m)\} = P\{S_m > g(m)\}$ and $p_m^+ = p_m$ for every m, irrespective of the properties of g. We now show that, regardless of the form of F, these equalities hold asymptotically as $m \rightarrow \infty$, i.e.,

$$(4.2.6) \qquad P\{S_m \geq g(m)\} \sim P\{S_m > g(m)\}$$

and

$$(4.2.7) \qquad p_m^+ \sim p_m,$$

assuming the hypotheses of Cramér's theorem (1.2.7) and Corollary 3.2.3, respectively.

The truth of (4.2.6) is not surprising, as (4.2.2) suggests the result

$$P\{S_m \geq g(m)\} \sim P\{Z \geq \tilde{\Psi}(m)\}$$

and Z has a continuous distribution. A straight-forward way of

proving (4.2.6) is to note that the proof of Cramér's theorem carries through as well for the non-strict crossing probabilities $P\{S_m \geq g(m)\}$ as for $P\{S_m > g(m)\}$. An alternative technique, to be used also in deducing (4.2.7) when Theorem 2.1.1 is in force, is to bound $\{S_m \geq g(m)\}$ according to

$$(4.2.8) \qquad \{S_m > g(m)\} \subseteq \{S_m \geq g(m)\} \subseteq \{S_m > g^-(m)\},$$

where g^- minorizes, but closely approximates, g. A suitable choice for g^- is

$$(4.2.9) \qquad g^-(t) = [1 - 1/\Psi^3(t)] \cdot g(t).$$

Clearly $\Psi^-(t) \sim \Psi(t) \to \infty$ as $t \to \infty$, where $g^-(t) \equiv t^{1/2}\Psi^-(t)$. Moreover, with f given by (2.4.23) and $\tilde{\Psi}^-(t) \equiv [tf(\Psi^-(t)/t^{1/2})]^{1/2}$ in accordance with (2.3.7),

$$\begin{aligned}
\tilde{\Psi}^2(t) - (\tilde{\Psi}^-(t))^2 &= t[f(\frac{\Psi(t)}{\sqrt{t}}) - f(\frac{\Psi^-(t)}{\sqrt{t}})] \\
&= (1 + o(1))t \cdot 2\frac{\Psi(t)}{\sqrt{t}}(\frac{\Psi(t)}{\sqrt{t}} - \frac{\Psi^-(t)}{\sqrt{t}}) \\
&= (1 + o(1)) \cdot 2\Psi(t)(\Psi(t) - \Psi^-(t)) \\
&= (1 + o(1)) \cdot 2/\Psi(t) = o(1)
\end{aligned}$$

as $t \to \infty$. From Cramér's theorem it follows that

$$(4.2.10) \qquad P\{S_m > g^-(m)\} \sim P\{S_m > g(m)\};$$

in light of (4.2.8), (4.2.6) holds.

Turning to (4.2.7), we consider separately the two results, Theorems 3.2.1 and 2.1.1, from which Corollary 3.2.3 follows. For Theorem 3.2.1 we rehash the proof; for Theorem 2.1.1 this method is direct but laborious and we choose to employ instead the approach of (4.2.8-9).

Theorem 3.2.1 is derived from the corresponding result, Theorem 3.1.1, for Brownian motion. But as a consequence of Ylvisaker's (1968) theorem,

$$\begin{aligned} &P\{B(t) \geq g(t) \quad \text{for some} \quad t \geq s\} \\ (4.2.11) \qquad &= P\{B(t) > g(t) \quad \text{for some} \quad t \geq s\} \end{aligned}$$

for every s, so Theorem 3.2.1 for p_m^+ follows in the same way as for p_m, and (4.2.7) holds.

Routine calculations show that the function g^- of (4.2.9) satisfies the hypotheses of Theorem 2.1.1, with the possible exception of the monotonicity of $g^-(t)/t$, and that $(g^-)'(t) \sim g'(t)$ and $(\Psi^-)'(t) \sim \Psi'(t)$ as $t \to \infty$. In addition, Lemma 2.2.1 continues to hold true -- some parts in negligibly weaker form, such as $(g^-)'(t) \leq (1 + o(1))g^-(t)/t$ for the second half of (b) -- and it is not hard to see that the conclusion (2.1.6) holds for g^-. Thus by (4.2.8) and (4.2.10),

$$p_m \leq p_m^+ \leq (1 + o(1)) \frac{(g^-)'(m)}{\sqrt{m}(\Psi^-)'(m)} P\{S_m > g^-(m)\}$$

$$= (1 + o(1)) \frac{g'(m)}{\sqrt{m}\Psi'(m)} P\{S_m > g(m)\}$$

$$= (1 + o(1))I_m = (1 + o(1))p_m,$$

and (4.2.7) holds.

Summarizing, when the assumptions of Corollary 3.2.3 are in force, the four weak-law probabilities

(4.2.12) $$P\{S_m > g(m)\}, \quad P\{S_{m+1} > g(m + 1)\},$$
$$P\{S_m \geq g(m)\}, \quad P\{S_{m+1} \geq g(m + 1)\}$$

are all asymptotically equivalent, as are (4.1.1-4).

The results are quite different when the hypotheses of Theorem 3.4.1 are assumed. In particular, it is immediately clear from (3.4.11-12) that (4.2.3-4) fail in this case. Furthermore, (4.2.6-7) are true if (3.4.9) holds but false if (3.4.10) holds. This can be verified by reviewing the proof of Theorem 3.4.1, but we'll introduce minorizing approximants to g and proceed as in (4.2.8).

Assume first that (3.4.9) holds. For $\delta > 0$ let

$$g_\delta(t) = \alpha - \delta + \epsilon t.$$

Apply (3.4.12) to both g and g_δ and use (4.2.8) (with $g^- = g_\delta$) to conclude

$$1 \leq \liminf_{m \to \infty} (P\{S_m \geq g(m)\}/P\{S_m > g(m)\})$$

$$\le \limsup_{m \to \infty} (P\{S_m \ge g(m)\}/P\{S_m > g(m)\}) \le \exp(|\theta_0|\delta).$$

Now let $\delta \to 0$ to get (4.2.6). Relation (4.2.7) is proved in the same way.

Approximations g_δ to g with ever increasing precision are not available in the lattice case (3.4.10), due to the restriction of α to integer multiples of $h =$ (span of F). In fact,

(4.2.13) $\quad P\{S_m \ge g(m)\} = P\{S_m > \alpha - h + \epsilon m\} \sim \exp(|\theta_0|h)P\{S_m > g(m)\}$

and similarly

(4.2.14) $\quad p_m^+ = P\{S_n > \alpha - h + \epsilon n \text{ for some } n \ge m\} \sim \exp(|\theta_0|h)p_m.$

4.3. Two boundaries.

The first task of this section is the

PROOF of Theorem 4.1.1. The right side of (4.1.6) overestimates the left by

$$P\{S_{n_1} > b(n_1) \text{ for some } n_1 \ge m,\ S_{n_2} < -a(n_2) \text{ for some } n_2 \ge m\}$$

(4.3.1)
$$\begin{aligned} &\le P\{S_p > b(p),\ S_q < -a(q) \\ &\qquad \text{for some } p, q \text{ satisfying } q > p \ge m\} \\ &\quad + P\{S_p < -a(p),\ S_q > b(q) \\ &\qquad \text{for some } p, q \text{ satisfying } q > p \ge m\}. \end{aligned}$$

Let us examine the first term, which equals

(4.3.2) $\quad \sum_{p=m}^{\infty} P\{S_n \leq b(n) \text{ for } m \leq n < p,\ S_p > b(p),\ S_q < -a(q)$

$\text{for some } q > p\}.$

The summand here is

$$P\{S_n \leq b(n) \text{ for } m \leq n < p,\ S_p > b(p)\}$$

times

$$\begin{aligned} &P\{S_q < -a(q) \text{ for some } q > p \mid S_n \leq b(n) \text{ for } m \leq n < p, \\ &\qquad\qquad\qquad S_p > b(p)\} \\ &\leq P\{S_q - S_p < -(a(q) + b(p)) \text{ for some} \\ &\qquad q > p \mid S_n \leq b(n) \text{ for } m \leq n < p,\ S_p > b(p)\} \\ &= P\{S_q - S_p < -(a(q) + b(p)) \text{ for some } q > p\} \\ &= P\{-S_k > a(k + p) + b(p) \text{ for some } k \geq 1\}; \end{aligned}$$

the sum (4.3.2) is therefore majorized by

$$\begin{aligned} &\sum_{p=m}^{\infty} P\{S_n \leq b(n) \text{ for } m \leq n < p,\ S_p > b(p)\} \\ &= P\{S_n > b(n) \text{ for some } n \geq m\} \end{aligned}$$

times

(4.3.3) $\quad P\{-S_k > a(k) + b(m) \text{ for some } k \geq 1\}.$

We shall show that the probabilities (4.3.3) decrease to 0 as

$m \uparrow \infty$, so that

$$\text{(first term on right in (4.3.1))}$$

$$= o(\text{first term on right in (4.1.6)}). \tag{4.3.4}$$

In just the same way

$$\text{(second term on right in (4.3.1))}$$

$$= o(\text{second term on right in (4.1.6)}) \tag{4.3.5}$$

and the proof of (4.1.6) is complete.

Since $b(m) \uparrow \infty$ as $m \uparrow \infty$ (as follows from $b \in U(S)$), the events in (4.3.3) decrease to

$$\bigcap_{n=1}^{\infty} \{-S_k > a(k) + n \quad \text{for some} \quad k \geq 1\} \tag{4.3.6}$$

Because $a \in U(-S)$,

$$P(\bigcup_{k^*=1}^{\infty} \{-S_k \leq a(k) \quad \text{for all} \quad k > k^*\}) = 1. \tag{4.3.7}$$

Hence the event (4.3.6) has the same probability as

$$\bigcup_{k^*=1}^{\infty} \bigcap_{n=1}^{\infty} \{-S_k > a(k) + n \quad \text{for some} \quad 1 \leq k \leq k^*\}$$

$$\bigcup_{k^*=1}^{\infty} \bigcap_{n=1}^{\infty} \{\max_{1 \leq k \leq k^*} (-S_k) > n\} = \emptyset,$$

namely, zero. $\square$

One might ask how the two terms on the right in (4.1.6) compare in size. We shall answer this question assuming that $a = b = g$ satisfies the hypotheses of Corollary 3.2.3, i.e., of either Theorem 3.2.1 or Theorem 2.1.1.

Recall that Theorem 3.2.1 is an invariance principle: the asymptotic rate at which the probabilities

$$p_m = P\{S_n > g(n) \quad \text{for some} \quad n \geq m\}$$

decrease to zero is the same for any random walk, including $(-S)$. Thus each term on the right in (4.1.6) equals $(1 + o(1))\, J_m$ as $m \to \infty$, where $J_{\cdot}$ is defined by (3.1.4).

In the case of Theorem 2.1.1 we need only to examine the corresponding weak-law probabilities using (1.2.7):

$$(4.3.8) \qquad P\{S_m > g(m)\} \sim P\{Z > \Psi(m)\}\exp[\Psi^2(m)\frac{\Psi(m)}{\sqrt{m}}\lambda(\frac{\Psi(m)}{\sqrt{m}})],$$

$$(4.3.9) \qquad P\{-S_m > g(m)\} \sim P\{Z > \Psi(m)\}\exp[-\Psi^2(m)\frac{\Psi(m)}{\sqrt{m}}\lambda(-\frac{\Psi(m)}{\sqrt{m}})],$$

(4.3.9) holding since $(-X)$ has Cramér series $(-\lambda(-(\cdot)))$. The ratio $(4.3.8)/(4.3.9)$ is

$$(4.3.10) \qquad r_m \equiv \exp\{\Psi^2(m)\frac{\Psi(m)}{\sqrt{m}}[\lambda(\frac{\Psi(m)}{\sqrt{m}}) + \lambda(-\frac{\Psi(m)}{\sqrt{m}})]\}.$$

In light of (2.1.6) r_m is also the ratio of the first term on the right in (4.1.6) to the second when $a = b = g$ satisfies the hypotheses of Theorem 2.1.1.

If $\Psi(m) = o(m^{1/6})$, i.e., if $g(m) = o(m^{2/3})$, then $r_m \to 1$, in accordance with out investigation for Theorem 3.2.1. Likewise, provided the constant term $\lambda_0 = EX^3/6$ in the Cramér series λ vanishes (in particular, if X has a symmetric distribution function F), then

$$r_m = \exp\{\Psi^2(m)\frac{\Psi(m)}{\sqrt{m}}[\lambda_1\frac{\Psi(m)}{\sqrt{m}} + \lambda_2(\frac{\Psi(m)}{\sqrt{m}})^2 - \lambda_1\frac{\Psi(m)}{\sqrt{m}} + \lambda_2(\frac{\Psi(m)}{\sqrt{m}})^2 + o(\frac{\Psi(m)}{\sqrt{m}})^2]\}$$

$$(4.3.11) \qquad \sim \exp\{2\lambda_2 \frac{\Psi^5(m)}{m^{3/2}}\}$$

tends to unity whenever $\Psi(m) = o(m^{3/10})$.

Suppose $\lambda_0 \neq 0$ and $m^{1/6} = o(\Psi(m))$. Then

$$(4.3.12) \qquad r_m \to \begin{matrix}\infty \\ 0\end{matrix} \text{ according as } \lambda_0 \begin{matrix} > \\ < \end{matrix} 0.$$

The result (4.3.12) agrees with intuition: If the distribution of the component X has a long right-hand tail, as is typically the case when $EX = 0$, $\mathrm{Var}\, X = 1$, and $EX^3 > 0$, large positive deviations for S are more likely than large negative deviations.

We can refine (4.3.11-12). For example, if $\Psi(m) = o(m^{3/10})$, then

$$(4.3.13) \qquad r_m = \exp\{2\lambda_0\frac{\Psi^3(m)}{\sqrt{m}} + O(\frac{\Psi^5(m)}{m^{3/2}})\},$$

regardless of the sign of λ_0. If $\Psi(m) = o(m^{5/14})$, then

$$(4.3.14) \qquad r_m \sim \exp\{2\lambda_0 \frac{\Psi^3(m)}{\sqrt{m}} + 2\lambda_2 \frac{\Psi^5(m)}{m^{3/2}} + O(\frac{\Psi^7(m)}{m^{5/2}})\},$$

and so on.

CHAPTER V

ASYMPTOTIC EXPANSIONS

5.1. Introduction.

As defined in Section 1.2, the problem of convergence rates related to the WLLN is to determine the probabilities

(5.1.1) $P\{S_m > g(m)\}$

up to a factor $(1 + o(1))$ as $m \to \infty$ when the random walk S satisfies the assumptions of Section 1.1 and the boundary g satisfies

(5.1.2) $\Psi(m) = \frac{g(m)}{\sqrt{m}} \to \infty$

as $m \to \infty$. A solution to this problem is by definition an approximation to (5.1.1) whose relative error tends to zero as $m \to \infty$. We can guarantee that this first-order approximation is of a specified quality by determining an asymptotic upper bound on the rate at which the relative error vanishes. The guaranteed accuracy of a second-order approximation whose relative error is known to vanish more quickly than this bound is greater. Iterating this idea we arrive at the problem of asymptotic expansions related to the WLLN: to develop an asymptotic expansion for (5.1.1) when (5.1.2) obtains. In Section 5.2 we present a solution in the Cramér case $\Psi(m) = o(m^{1/2})$,

$\Psi(m) \to \infty$. The solution in the WLLN case $\Psi(m) = cm^{1/2}$ is due to Bahadur and Ranga Rao (1960) and is omitted here.

The problem of asymptotic expansions related to the SLLN is to derive an asymptotic expansion for the probabilities

(5.1.3) $$p_m = P\{S_n > g(n) \text{ for some } n \geq m\}$$

when $g \in U(S)$. The dominant term in such an expansion is given for boundaries $\Psi(t) = g(t)/t^{1/2}$ of slow, moderate, and rapid growth in Theorems 3.2.1, 2.1.1, and 3.4.1, respectively, the slow and moderate ranges overlapping considerably. In Section 5.3 we obtain a partial asymptotic expansion of p_m for boundaries of slow growth. This expansion is of higher order the more slowly Ψ increases and in fact is complete, i.e., of infinite order, when Ψ is not also in the range of moderate growth. In Section 5.4 we derive an asymptotic upper bound on the relative error in the approximation of Theorem 2.1.1 for boundaries of moderate growth. The problems of finding even the second term in an asymptotic expansion of p_m for Ψ in the moderate range and of bounding the error in the approximation of Theorem 3.4.1 remain open.

5.2. An asymptotic expansion related to the WLLN.

Throughout this section we assume (5.1.2).

We begin with an examination of the case $F = \Phi$ of standard normal components. In this case the probability (5.1.1) equals $P\{Z > \Psi(m)\}$, and the solution to the convergence rate problem is given most simply by (1.2.8). The solution to the asymptotic expansion problem is equally well known (Feller, 1968, p. 193): for any $n \geq 0$

$$(5.2.1) \qquad P\{Z > \Psi(m)\} = [\sqrt{2\pi}\Psi(m)]^{-1}\exp[-\tfrac{1}{2}\Psi^2(m)]$$

$$\cdot \left[1 - \frac{1}{\Psi^2(m)} + \frac{1\cdot 3}{\Psi^4(m)} - \frac{1\cdot 3\cdot 5}{\Psi^6(m)} + \cdots + (-1)^n \frac{1\cdot 3\cdot \cdots \cdot(2n-1)}{\Psi^{2n}(m)} + o\left(\frac{1}{\Psi^{2n}(m)}\right)\right]$$

as $m \to \infty$.

For general F the solution presented here to the asymptotic expansion problem in the Cramér case

$$(5.2.2) \qquad \Psi(m) = o(\sqrt{m})$$

is due to Saulis (1969); see also Ibragimov and Petrov (1971, p. 416).

In addition to the assumptions of Section 1.1, suppose that F has characteristic function $E\exp(itX)$ satisfying

$$(5.2.3) \qquad \limsup_{|t| \to \infty} |Ee^{itX}| < 1,$$

the so-called Cramér's condition (C). The condition (5.2.3) holds, in particular, if F has a nonzero absolutely continuous component and fails if F is a lattice distribution. Then for any $n \geq 0$

$$(5.2.4) \qquad P\{S_m > g(m)\} = P\{Z > \Psi(m)\} \cdot \exp[\Psi^2(m)\frac{\Psi(m)}{\sqrt{m}}\lambda(\frac{\Psi(m)}{\sqrt{m}})]$$

$$\cdot\ [\sum_{k=0}^{n} Q_k(m) + o((\frac{\Psi(m)}{\sqrt{m}})^n)]$$

as $m \to \infty$, where the functions $Q_0, Q_1, \ldots$ satisfy

$$(5.2.5) \qquad Q_k(m) = O((\frac{\Psi(m)}{\sqrt{m}})^k), \quad k \geq 0.$$

Saulis (1969) gives explicit formulas for the functions Q_k. In particular,

$$(5.2.6a) \qquad Q_0(m) = 1,$$

reducing (5.2.4) to (1.2.7) in the case $n = 0$. For $n = 1$ the approximation (5.2.4) fares better than is guaranteed by (5.2.5): $Q_1(m) \equiv 0$ if $\lambda_0 = 0$ and

$$Q_1(m) = \frac{\lambda_0}{\sqrt{m}} \cdot [\frac{\Psi^2(m) - 1}{\sqrt{2\pi}P\{Z > \Psi(m)\}} e^{-\frac{1}{2}\Psi^2(m)} - \Psi^3(m)]$$

$$(5.2.6b) \qquad = -(1 + o(1))\ 3\lambda_0 \frac{1}{\Psi(m)\sqrt{m}}$$

otherwise.

In the language of asymptotic series, $\sum_k Q_k(m)$ is an asymptotic expansion for

$$\frac{P\{S_m > g(m)\}}{P\{Z > \Psi(m)\} \cdot \exp[\Psi^2(m)\frac{\Psi(m)}{\sqrt{m}}\lambda(\frac{\Psi(m)}{\sqrt{m}})]}$$

as $m \to \infty$ with respect to the auxiliary asymptotic sequence $((\Psi(m)/m^{1/2})^n)$.

The normal probability on the right in (5.2.4) can be expanded using (5.2.1). The power series representation of λ yields an asymptotic expansion for the argument of the exponential factor on the right in (5.2.4). An expansion for that factor can then be obtained if $g(m) = o(m^{1 - \eta})$ for some $\eta > 0$ by employing the Taylor's series for the exponential function.

Recall from (1.2.9a) that the probabilities $P\{S_m > g(m)\}$ and $P\{Z > \Psi(m)\}$ agree up to a factor $(1 + o(1))$ whenever $g(m) = o(m^{2/3})$. More generally, as seen below, if

(5.2.7) $\quad g(m) = o(m^{1 - \eta})$

with

(5.2.8) $\quad \frac{1}{3} \le \eta < \frac{1}{2},$

then the invariance principle

(5.2.9) $\quad P\{S_m > g(m)\} = (\text{right side of } (5.2.1)) \quad \text{as} \quad m \to \infty$

obtains, provided

(5.2.10) $\quad n \le \frac{3}{2} \frac{\eta - 1/3}{1/2 - \eta}$

With $\eta = 1/3$, as in (1.2.9a), the bound on n is 0, and (5.2.9) reduces to (1.2.9a). As η increases to $1/2$, the bound on n

increases to ∞. The inequality (5.2.10) can be inverted:

(5.2.10a) $\eta \geq \frac{n+1}{2n+3}$.

For example, to apply (5.2.9) with $n = 1$ we must have $\eta \geq 2/5$, i.e., $g(m) = o(m^{3/5})$. For $n = 2$ we require $g(m) = o(m^{4/7})$, and so on. If (5.2.7) holds for every $\eta < 1/2$, e.g., if

(5.2.11) $g(m) = \sqrt{m} \cdot (\log m)^{\beta}, \quad \beta > 0,$

then (5.2.9) provides a complete asymptotic expansion for m: (5.2.9) holds for each $n \geq 0$.

The proof of (5.2.9) from (5.2.10) is easy. Since $g(m) = o(m^{2/3})$, i.e., $\Psi(m) = o(m^{1/6})$, (5.2.4) with $n = 1$ yields

$$P\{S_m > g(m)\} = P\{Z > \Psi(m)\} \cdot [1 + O(\frac{\Psi^3(m)}{\sqrt{m}})] \cdot [1 + O(\frac{\Psi(m)}{\sqrt{m}})]$$

$$= P\{Z > \Psi(m)\} \cdot [1 + O(\frac{\Psi^3(m)}{\sqrt{m}})]$$

using (5.1.2). So (5.2.9) holds if

$$\frac{\Psi^3(m)}{\sqrt{m}} = o(\frac{1}{\Psi^{2n}(m)})$$

i.e., if

$$g(m) = o(m^{\frac{n+2}{2n+3}}).$$

For this it is enough that $1 - \eta \leq (n + 2)/(2n + 3)$, i.e., that (5.2.10a) hold.

5.3. An asymptotic expansion related to the SLLN.

The following theorem (Wichura, 1980) gives a complete expansion for boundary crossing probabilities for Brownian motion.

THEOREM 5.3.1 (Wichura). Suppose the positive function $g \in U(B)$ satisfies the growth condition

$$(5.3.1) \qquad \frac{g(t)}{t^{\zeta}} \uparrow$$

as $t \uparrow$ for some $\zeta > 0$. Let $n \geq 0$, and suppose g has $(n + 1)$ continuous derivatives satisfying the smoothness condition

$$(5.3.2) \qquad g^{(k)}(u) \sim g^{(k)}(t) \quad \text{as} \quad u \sim t \to \infty, \quad 1 \leq k \leq n + 1,$$

and the growth condition

$$(5.3.3) \qquad g^{(k)}(t) \asymp t^{1/2 - k}\psi(t), \quad 1 \leq k \leq n + 1.$$

Finally, suppose

$$(5.3.4) \qquad P\{B(t) > g(t) \ \text{ for some } \ t \geq s\} = o\left(\frac{1}{\psi^{2n}(s)}\right)$$

as $s \to \infty$. Then there exist functions $A_0, A_1, \ldots$ satisfying

$$(5.3.5) \qquad A_k(t) = O\left(\frac{1}{\sqrt{t}\,\psi^{2k - 1}(t)}\right), \quad 0 \leq k \leq n,$$

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as $t \to \infty$ such that with

$$(5.3.6) \qquad J_k(s) \equiv \int_s^\infty \frac{1}{2}\frac{1}{\sqrt{2\pi t}}\, e^{-\frac{1}{2}\psi^2(t)}\, A_k(t)\, dt$$

we have

$$P\{B(t) > g(t) \text{ for some } t \geq s\}$$

$$(5.3.7) \qquad = \sum_{k=0}^{n} J_k(s) + o\left(\int_s^\infty \frac{1}{\sqrt{t}}\, e^{-\frac{1}{2}\psi^2(t)} \frac{1}{\sqrt{t}\,\psi^{2n-1}(t)}\, dt\right)$$

as $s \to \infty$. ▯

REMARK 5.3.2. (a) Explicit definitions of A_k are given by Wichura (1980); for example,

$$(5.3.8a) \qquad A_0 = 2g',$$

$$(5.3.8b) \qquad A_1 = \frac{g''}{(g')^2},$$

$$(5.3.8c) \qquad A_2 = -\frac{4(g'')^2}{(g')^5} + \frac{g'''}{(g')^4}.$$

In particular, the theorem reduces to Theorem 3.1.1 in the case $n = 0$.

(b) Theorem 3.1.1 is of help in checking (5.3.4). ▯

As Theorem 3.2.1 follows from Theorem 3.1.1, so Theorem 5.3.1 yields the invariance principle

THEOREM 5.3.3. Let S satisfy the assumptions of Section 1.1. Suppose the positive function $g \in U$ satisfies the growth conditions

(5.3.9) $$\frac{g(t)}{t^{1/2}} \uparrow$$

and

(5.3.10) $$\frac{g(t)}{t^{1-\eta}} \downarrow 0$$

where

(5.3.11) $$\frac{1}{3} \le \eta < \frac{1}{2}.$$

Let $n \geq 0$, and suppose g has $(n+1)$ continuous derivatives satisfying the smoothness condition (5.3.2) and the growth condition (5.3.3). Then with A_k and $J_k(\cdot)$ as in Theorem 5.3.1 and with p_m given as usual by (5.1.3),

(5.3.12) $$p_m = \sum_{k=0}^{n} J_k(m) + o\left(\int_m^\infty \frac{1}{\sqrt{t}}\, e^{-\frac{1}{2}\psi^2(t)} \frac{1}{\sqrt{t}\,\psi^{2n-1}(t)}\, dt\right)$$

as $m \to \infty$, provided (5.2.10) holds. ▯

REMARK 5.3.4. (a) The derivatives of g can be computed ignoring ψ, in the sense of (5.3.3), for all boundaries g of practical interest (e.g., for g as in (1.3.9) or Example 1.6.3). Condition (5.3.2) is likewise a mild restriction.

(b) As always, the regularity conditions for g need only be satisfied for large t.

(c) The smoothness condition (5.3.2) can be weakened to

$$(5.3.13) \qquad g^{(k)}(u) \sim g^{(k)}(t) \quad \text{when} \quad t, u \to \infty \quad \text{as in (2.1.5)}$$

provided (3.2.4-6) hold.

(d) Theorem 5.3.3 is clearly the strong-law counterpart to the result (5.2.9) of the previous section. With $\eta = 1/3$ the bound on n in (5.2.10) is 0, and Theorem 5.3.3 reduces to Theorem 3.2.1. In a case such as (5.2.11), for each $\eta < 1/2$ the boundary g satisfies (5.3.10) for all large t, and so the expansion (5.3.12) is complete.

(e) Suppose, in addition to the assumptions of Theorem 5.3.3, that the boundary is also in the range of moderate growth, i.e., that (2.1.3) holds for some $0 < \delta < 1/6$. In this case one can show without difficulty that

$$(5.3.14) \qquad \int_m^\infty \frac{1}{\sqrt{t}}\, e^{-\frac{1}{2}\psi^2(t)} \frac{1}{\sqrt{t}\,\psi^{2k-1}(t)}\, dt \approx \frac{p_m}{\psi^{2k}(m)} \approx \frac{P\{S_m > g(m)\}}{\psi^{2k}(m)}, \quad k \geq 0.$$

Provided that (5.3.5) is also a lower bound, i.e., that

$$(5.3.15) \qquad A_k(t) \approx \frac{1}{\sqrt{t}\,\psi^{2k-1}(t)} \quad \text{as} \quad t \to \infty,$$

the k th term $J_k(m)$ in (5.3.12) is also of order of magnitude (5.3.14). For $k = 1$ one can even show that

$$(5.3.16) \quad J_1(m) - \frac{1}{2}\Lambda_m p_m \approx \frac{p_m}{\psi^2(m)}$$

where (cf. (5.3.8b))

$$(5.3.17) \quad \Lambda_m \equiv \frac{A_1(m)}{g'(m)} = \frac{g''(m)}{(g'(m))^3} \approx \frac{1}{\psi^2(m)}.$$

In particular, if $\eta \geq 2/5$ in (5.3.10), i.e., if $g(t)/t^{3/5} \downarrow 0$, then

$$(5.3.18) \quad p_m = [1 + \frac{1}{2}\Lambda_m + o(\frac{1}{\psi^2(m)})]J_0(m),$$

demonstrating that the relative error in the approximation $J_m \equiv J_0(m)$ of Theorem 3.2.1 is in this case of the same order of magnitude as $1/\psi^2(m)$. $\square$

To prove Theorem 5.3.3 one uses Theorem 5.3.1 and the invariance principle of Lemma 3.2.5 with

$$(5.3.19) \quad \alpha = 2 - \frac{1}{1 - \eta}.$$

The particulars of the proof require detailed knowledge of the functions A_k of (5.3.5) and are omitted.

5.4. An asymptotic upper bound on the relative error in (2.1.6).

The following theorem is a refinement of Theorem 2.1.1.

THEOREM 5.4.1. Suppose, in addition to the hypotheses of Theorem 2.1.1, that g is twice differentiable with

$$(5.4.1) \qquad g''(t) = O\left(\frac{g(t)}{t^2}\right) \quad \text{as} \quad t \to \infty .$$

Then we have the refinement

$$(5.4.2) \qquad p_m = \left[1 + O\left(\frac{\log \Psi^2(m)}{\Psi^2(m)}\right) + O\left(\frac{\Psi(m)}{\sqrt{m}}\log\left(\frac{1}{\Psi(m)/\sqrt{m}}\right)\right)\right] I_m$$

of (2.1.6). ▯

REMARK 5.4.2. (a) The growth condition (5.4.1) is satisfied in particular if the derivatives of g can be computed ignoring Ψ in the sense of (5.3.3).

(b) The two-term big-oh bound of (5.4.2) on the relative error in the approximation I_m for p_m reduces to the first term when $g(m) = O(m^{2/3})$ and to the second in the contrasting case $m^{2/3} = O(g(m))$.

(c) For comparison, recall that the relative errors of the approximations (1.2.7) and (1.2.8) are $O(\Psi(m)/m^{1/2})$ and $O(1/\Psi^2(m))$, respectively. For (1.2.7) we are ignoring the good fortune of (5.2.6b).

(d) In case $g(t)/t^{3/5} \nrightarrow 0$, the approximation J_m of Theorem 3.2.1 has greater accuracy (relative error $\approx 1/\Psi^2(m)$) than that guaranteed by (5.4.2) (relative error

$= O((\log \Psi^2(m))/\Psi^2(m)))$ for I_m. See Remark 5.3.4(e). ▯

PROOF of Theorem 5.4.1. As a consequence of Theorem 2.1.1, whose proof we shall review in order to sketch that of (5.4.2),

$$(5.4.3) \qquad p_m \approx [\Psi(m)]^{-1}\exp[-\tfrac{1}{2}\tilde{\Psi}^2(m)] \quad \text{as} \quad m \to \infty$$

with $\tilde{\Psi}$ defined by (2.3.7). It follows as in Section 2.3 that if the definition (2.3.2) is replaced by

$$(5.4.4) \qquad v_m = \lceil m(1 + \frac{1}{\delta^*}\frac{\log \Psi^2(m)}{\Psi^2(m)})\rceil$$

with $0 < \delta^* < \delta$, then (2.3.3-4) continue to hold, and

$$(5.4.5) \qquad p_{v_m} = O(\frac{1}{\Psi^2(m)}) \cdot p_m \quad \text{as} \quad m \to \infty .$$

Assuming (5.4.1), the definition of $\mathcal{C}_m$ (see Section 2.4) and Taylor's theorem imply the improvement

$$(5.4.6) \qquad \mathcal{C}_m = (1 + O(\frac{\log \Psi^2(m)}{\Psi^2(m)})) \cdot g'(m) \quad \text{as} \quad m \to \infty$$

to (2.4.1).

Now the proof of Lemma 2.4.1 shows that the integral in (2.5.27) is smaller than 1 by an amount $O(\mathcal{C}_m{}^{1/2}) = O((\Psi(m)/m^{1/2})^{1/2})$. But Lorden's (1970) method can be used to show that for any $\xi > 0$

$$\exp(\xi E_{m,\theta_1(m)} R_y) \le E_{m,\theta_1(m)} \exp(\xi R_y)$$

(5.4.7) $$\le \frac{E_{m,\theta_1(m)}[(\xi X^+ + 1)\exp(\xi X^+)] - 1}{\xi E_{m,\theta_1(m)} X} .$$

With ξ chosen within the interval of finiteness of the cgf K, this leads to the improved estimate

$$1 - O(\frac{\Psi(m)}{\sqrt{m}} \log(\frac{1}{\Psi(m)/\sqrt{m}}))$$

for the final integral in (2.5.27). The relation (2.5.30) can be developed into a complete asymptotic expansion; in particular, one can show that the relative error in (2.5.30) is $O(1/\Psi^2(m)) + O(\Psi(m)/m^{1/2})$. The same two-term bound on the error holds for (2.4.9) and hence for (2.4.12). As (2.3.1) can be tightened to (5.4.5), so (2.4.13) can be improved to

(5.4.8) $$p_{v_m}(\ell_m) = O(\frac{1}{\Psi^2(m)}) \cdot p_m \quad \text{as} \quad m \to \infty .$$

Combining the various bounds we arrive at (5.4.2). $\square$

<u>REMARK</u> 5.4.3. To get further terms in an asymptotic expansion for p_m in the case of Theorem 2.1.1, one would need to approximate g more closely than by the straight lines $\underline{\ell}_m$ and $\overline{\ell}_m$ of Section 2.4, perhaps by higher-order polynomials. One would then seek appropriate results from nonlinear renewal theory to replace the use of Lorden's (1970) bound. $\square$

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Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER No. 14	2. GOVT ACCESSION NO. AD-A102	3. RECIPIENT'S CATALOG NUMBER 070
4. TITLE (and Subtitle) CONVERGENCE RATES RELATED TO THE STRONG LAW OF LARGE NUMBERS		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) James Allen Fill		8. CONTRACT OR GRANT NUMBER(s) N00014-77-C-0306
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics Stanford University Stanford, California		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS (NR-042-373)
11. CONTROLLING OFFICE NAME AND ADDRESS Statistics & Probability Program (Code 436) Office of Naval Research Arlington, Virginia 22217		12. REPORT DATE December 15, 1980
		13. NUMBER OF PAGES 99
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for Public Release and Sale: Distribution Unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Also supported in part by National Science Foundation Grants SOC 72-05228 A04 and MCS 72-04364 A04 from University of Chicago.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Random walk, laws of large numbers, convergence rates, boundary crossing probabilities, invariance principles, large deviations, law of iterated logarithm, Brownian motion.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let $(S_n)_{n\geq 0}$ be a random walk whose summands are i.i.d. F and have zero mean, unit variance, and finite moment generating function. We determine $p_m \equiv P\{\frac{S_n}{n} > c_n \text{ for some } n \geq m\}$ up to asymptotic equivalence for a wide range of boundaries (c_n). Our solution depends on an ever increasing number of terms in the Cramér series for F the more slowly c_n converges to zero.

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